Circuit-equivalent Models for Current-controlled Inverters

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Abstract—In this paper, we introduce a method to analyze three-phase inverters with current control as equivalent circuits. In contrast to existing methods, both the averaged power stage model and its closed-loop controller are represented as a single unified circuit. Since the complete system can be examined as one equivalent circuit, we can glean several insights on design and operation; particularly relating to the time scales at which the controller tracks reference signals and the ability to reject disturbances. We leverage the insights afforded by this general approach to outline design strategies for controllers in both synchronous and stationary reference frames.

I. INTRODUCTION

Three-phase dc-ac inverters are ubiquitous across a variety of energy-conversion and grid applications. State-of-the-art approaches for modeling dc-ac inverters are mainly focused on averaged power-stage models and their linearized representations for small-signal analysis [1]. In this paper, we propose a novel framework to model and analyze the control- and averaged physical-layer dynamics of inverters as a single equivalent circuit. In essence, we show that well-known control strategies can be equivalently understood as instances of circuit laws. This allows us to recast the closed-loop system as an equivalent circuit composed of elementary components (e.g., individual passive RLC elements, current sources, voltage sources) and apply a circuit-driven approach to control design. The proposed approach is intuitive, and it yields several insights that facilitate controller design. We also anticipate the unified circuit-based description would facilitate analysis (e.g., stability analysis of networks of inverters).

Related to our approach are impedance-based modeling methods. Under such frameworks, the inverter output terminal is modeled as an ideal source in conjunction with an equivalent impedance whose characteristics are based on the linearized control loops [2]–[5]. Although this method is related to circuit analysis, models in this setting do not represent the control loops explicitly as circuits. Hence, a circuit model with elementary constituent components is not realized. Also related to this work are analog control methods for inverters [6]. While these controllers are implemented with op-amp circuits, our approach is principally different since it is geared towards modeling and accounts for the representation of the full closed-loop system. Finally, we bring to attention control strategies that involve inverters emulating dynamics of nonlinear oscillator circuits [7] and synchronous machines [8] that innately yield equivalent-circuit representations.

For the case of current-controlled three-phase inverters, we demonstrate how algebraic relationships inherent in feedback loops and feedforward paths are translated to equivalent-circuits via elementary circuit laws. Our work differs from some of the related methods referenced above in the sense that the component-level structure of the equivalent circuit is laid bare and its realization does not require linearization. This modeling formalism is useful since it eliminates the boundary that has hitherto separated control- and physical-layer dynamics. Hence, the relationships between the various control signals and physical variables are clearly illustrated and this allow one to apply circuit-inspired rules of thumb to design. Along these lines, we first show how proportional-integral (PI) and proportional-resonant (PR) compensators, as commonly seen in the synchronous dq and stationary αβ frames, respectively, take the form of RLC circuits. Once this is established, it becomes apparent that the tuning of such controllers is tantamount to a circuit design problem where the time constants of the various circuits are to be systematically determined based on the desired closed-loop performance. This is in contrast to established approaches in the dq [1], [9], [10] and αβ [11]–[13] frames where design is transfer-function-based and physical intuition is lost. In summary, our paper provides the following contributions: i) we show that current-control feedback and feedforward loops intrinsically embed Kirchhoff’s Laws within them and leverage this foundational tenet to formulate a unified circuit model for three-phase inverters, ii) within the context of current control for inverters, we define a one-to-one consistency between circuit laws and classical control laws, and iii) we propose a circuit-driven design methodology for inverters in both the dq and αβ frames building on circuit-theoretic principles.

The remainder of this paper is organized as follows. Circuit-equivalent modeling basics are introduced in Section II. Models for the synchronous dq and the stationary αβ frame are provided in Section III. Numerical simulation results are provided in Section V. Concluding remarks are in Section V.

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II. Circuit Modeling Primer

To familiarize the reader with the circuit-equivalent model concept, we begin with the half bridge converter with current control in Fig. 1(a). An output filter with inductance $L$ and parasitic resistance $R$ interfaces the converter output with the network. The voltage $v_o$ generically represents the voltage across any active load, passive load, stiff voltage source, or any combination thereof. The output current, $i$, is measured and compared to the reference $i^*$. The error, $e := i^* - i$, is fed to a compensator, $G_c$, which produces the control effort $u$. Subsequently, the output voltage, $v_o$, is used as a feedforward signal which is added to $u$ to produce the terminal voltage command $v_t^*$. Lastly, the terminal command is scaled by the dc-side input voltage, $v_{dc}$, and processed by pulse width modulation (PWM) such that the converter terminal voltage, $v_t$, tracks $v_t^*$.

From here forward, we consider switch-cycle-averaged quantities such that PWM is abstracted away and $v_t = v_t^*$. Reflecting on the well-known structure in Fig. 1(a), it is evident that the compensator translates current differences into a voltage signal $u$. Hence, the input and output sides of the control block-diagram can be construed as instances of Kirchhoff’s current and voltage laws, respectively. Since $G_c$ translates a current signal into a voltage signal, it follows that the compensator itself is an impedance which we denote interchangeably by $z_c$. With due regard to Kirchhoff’s laws in Fig. 1(a), we arrive at the equivalent circuit in Fig. 1(b) where the compensator is represented as a passive $RLC$ impedance, $i^*$ translates to a current source, and the feedforward manipulates a controllable voltage source.

To illustrate how the compensator drives $i \rightarrow i^*$, we consider a controller with proportional, integral, derivative, and resonant terms with the following transfer function

$$G_c(s) = z_c(s) = k_p + k_i \frac{1}{s} + k_d s + \frac{k_r}{s^2 + \omega_c^2}. \tag{1}$$

This general controller is meant to highlight various applications where $i^*$ could have dc as well as ac components. Simpler PI and PR implementations are straightforwardly recovered by discarding unneeded terms. Comparison of (1) to basic circuit laws allows us to obtain the passive $RLC$ instantiation of $z_c$ in Fig. 1(b) where the proportional, integral, derivative, and resonant terms correspond to a resistor, capacitor, inductor, and resonant tank which are cascaded to obtain the controller. In the circuit setting, PI and PR controllers are obtained by short-circuiting unused components.

Reflecting further on the equivalent circuit model, it is apparent that the error current, $e$, flows through $z_c$ and the control effort, $u$, is the voltage across $z_c$. Here, ideal reference tracking at some frequency $\omega^*$ is obtained when $\|z_c(j\omega^*)\| \rightarrow \infty$ such that $z_c$ behaves as an open circuit and the only available path for the reference current is through the physical branch (i.e., $i \rightarrow i^*$). This is obtained at dc for $\omega^* = 0$ with a capacitive element, at some resonant frequency $\omega^* = \omega_r$ with an $LC$ tank, at high frequencies $\omega^* \rightarrow \infty$ with an inductor, and proportional control action maps straightforwardly as a voltage to the compensator, inductor, and resonant tank which are cascaded to obtain generators

Our study is restricted to balanced ac systems where the zero-axis component is discarded. Phase-locked-loop (PLL) and dc-side dynamics are also neglected and reserved for future work.

III. Circuit-Equivalent Models for Three-phase Inverters

We now apply the basic concepts sketched out in Section II to derive circuit-equivalent representations for inverters in the $dq$- and $\alpha\beta$-reference frames.

A. The Synchronous $dq$ Reference Frame

We begin with the well-established model in Fig. 2(a) which depicts the cross-coupling between axes, the $d$- and $q$-axis control loops with PI compensators, and feedforward signals. Application of the concepts in Section II allow us to translate

Figure 1: A half-bridge circuit with current control is depicted in (a). Kirchhoff’s current and voltage laws are embedded within the input and output sides of the compensator and allow us to obtain the equivalent closed-loop circuit model in (b).

resistor. With the circuit model, it is also clear that feedforward cancels voltage disturbances on the output of the converter.

In the remainder of the paper, we apply the fundamental concepts illustrated in Fig. 1 to three-phase inverters. All analysis will be carried out in two-dimensional reference frames where $dq$ and $\alpha\beta$ pertain to the synchronous and stationary reference frames, respectively. We adopt the short hand notation $x^{dq}$ and $x^{\alpha\beta}$ to denote the following complex quantities:

$$x^{dq} = x^d + jx^q, \quad x^{\alpha\beta} = x^\alpha + jx^\beta. \tag{2}$$

Our study is restricted to balanced ac systems where the zero-axis component is discarded. Phase-locked-loop (PLL) and dc-side dynamics are also neglected and reserved for future work.
this into the circuit representation in Fig. 2(b) where the PI compensator is an RC circuit, feedforward signals manipulate a voltage source, and the references are current sources. For convenience, we represent both axes compactly in a single circuit which admits complex-valued signals.

Unlike conventional transfer-function-based analysis where derivation of the closed-loop response requires several algebraic steps, the equivalent circuit gives this directly via the current divider rule. Assuming ideal sensing and feedforward cancellation of the \( v_d^{\text{dq}} \pm \omega L v_t^{\text{dq}} \) disturbance, the closed-loop response is

\[
H(s) = \frac{i^{\text{dq}}(s)}{u^{\text{dq}}(s)} = \frac{z_c(s)}{z_c(s) + z_t(s)} = \frac{k_p + k_1/s}{(k_p + k_1/s) + (R + sL)}. \tag{3}
\]

The above formulation matches what is well-known in the literature [1]. The key difference being that we arrive at it via the application of a well-known circuit law and obtained it directly by inspection of a circuit. Now consider the control-synthesis problem where a typical objective is to determine the PI control gains, \( k_p \) and \( k_1 \), such that \( H(s) \) is a first-order system with time constant \( \tau \). To this end, we highlight the following result.

**Theorem.** The closed-loop transfer function, \( H(s) \), is first-order with time constant, \( \tau \),

\[
H(s) = \frac{1}{1 + \tau s}, \tag{4}
\]

if and only if the time constant \( \tau_c \) of the RC circuit-equivalent of the PI compensator,

\[
\tau_c = \frac{k_p}{k_1}, \tag{5}
\]

matches the time constant \( \tau_t \) of the inductive output filter:

\[
\tau_t = \frac{L}{R}. \tag{6}
\]

**Proof.** To prove the forward direction, we begin by expressing

\[
H(s) = \frac{k_p s + k_1}{L s^2 + (R + k_p) s + k_1} = \frac{N(s)}{D(s)} \tag{7}
\]

For \( H(s) \) (as expressed above) to match the desired first-order system representation in (4) with time constant \( \tau \), we need

\[
D(s) = N(s)(1 + \tau s). \tag{8}
\]

Matching coefficients, we can conclude that this implies \( k_p = L/\tau \) and \( k_1 = R/\tau \), or equivalently, that \( \tau_c = \tau_t \). To prove the reverse direction, with elementary algebraic operations, we can express

\[
H(s) = \left(1 + \frac{1 + \tau_t s}{1 + \tau_c s} \cdot \frac{R}{k_1} \right)^{-1}. \tag{9}
\]

From above, we can infer that if \( \tau_c = \tau_t \), then \( H(s) \) is a first-order transfer function with time constant \( \tau = R/k_1 \).

A physical interpretation of the result above naturally emerges from the circuit-based model. In summary, it is a matching of the time-constants between each set of pairwise RC and RL circuit elements which gives a first-order circuit response. This interpretation is indeed qualitatively different from the frequency-domain notion of pole-zero cancellation in the closed-loop transfer function.

**B. The Stationary \( \alpha\beta \) Reference Frame**

The classical current-controlled inverter model in the \( \alpha\beta \) reference frame is illustrated in Fig. 3(a). Again, we apply the basic principles in Section II to arrive at the equivalent circuit model in Fig. 3(b). Since the reference signal and load voltage are sinusoidal, we adopt the typical strategy where the compensator takes on a PR form given by

\[
G_c(s) = z_c(s) = k_p + \frac{k_r s}{s^2 + \omega_r^2}, \tag{10}
\]

where the compensator resonant frequency, denoted as \( \omega_r \), is chosen to coincide with the load (or grid) frequency (e.g., 50 or 60 Hz). Again, the resonant structure revealed in Fig. 3(b) clearly shows that \( z_c \) acts as a high impedance path at the resonant frequency.

Mirroring (3), the relationship between the current reference, \( i^{\alpha\beta}\ast \), and output current, \( i^{\alpha\beta} \), is captured by the transfer function:

\[
H(s) = \frac{i^{\alpha\beta}(s)}{i^{\alpha\beta}\ast(s)} = \frac{z_c(s)}{z_c(s) + z_t(s)} = \frac{k_p (s^2 + \omega_r^2) + k_r s}{(s^2 + \omega_r^2)(sL + R) + k_p (s^2 + \omega_r^2) + k_r s}. \tag{11}
\]
Note that $H(s)$ is third-order and is generally difficult to analyze. However, the circuit-equivalent provides insights that are leveraged to facilitate design. The PR compensator admittance $y_c := z_c^{-1}$ is given by:

$$y_c = \frac{1}{z_c} = \frac{1}{k_p} \frac{s^2 + \omega_r^2}{s^2 + \frac{k_p}{k_c} s + \omega_r^2}. \tag{12}$$

Note that $y_c$ has the same form as a notch filter (i.e., bandstop filter) [14] with center frequency $\omega_r$. Furthermore, the ratio $k_r/k_p$ is related to the notch filter damping factor, $\zeta$, and resonant frequency, $\omega_r$, as follows:

$$\frac{k_r}{k_p} = 2\zeta\omega_r = \frac{2}{\tau_c}, \tag{13}$$

where $\tau_c$ is the time-constant of the compensator $RLC$ branch. This is consistent with classical dynamical systems analysis where second order linear systems are known to have a time constant equal to $\zeta\omega_r$. Rearranging terms in (11) allows us to isolate the dynamics of a notch filter, and yields

$$H(s) = \frac{k_p}{s^2 + \frac{k_p}{k_c} s + \omega_r^2}. \tag{14}$$

Now we proceed to the design task and first select the proportional gain, $k_p$. Since the notch factor in (14) is in-effect inactive at frequencies sufficiently far from $\omega_r$, the closed-loop response can be approximated as

$$H(s) \approx \frac{k_p}{sL + R + k_p} = \frac{k_p}{k_p + R} \left(1 + \frac{L}{k_p + R}s\right)^{-1}, \tag{15}$$

at those frequencies. With the approximate first-order behavior in (15), we can engineer the response to have a user-defined cut-off frequency, denoted as $\omega_c$, by choosing

$$k_p = \omega_c L - R, \tag{16}$$

where $\omega_c$ is sufficiently higher than $\omega_r$. Here the circuit interpretation reveals that the compensator resistance, $k_p$, is equal to the difference between the physical-branch inductive reactance and resistance at $\omega_c$.

We now shift focus to the resonant gain $k_r$. Mimicking the strategy used in the $dq$-domain above, we match the time constant of the second order compensator (as defined in (13)) with the time-constant of the physical $RL$ branch. This gives

$$\tau_c = 2\zeta\omega_r = \frac{L}{k_r}, \tag{17}$$

Substituting (16) into (17) yields the resonant gain

$$k_r = 2R\frac{k_p}{L} = 2\omega_c R - 2R^2 L. \tag{18}$$

This completes the design procedure for the PR controller. Here we note that the circuit-based interpretation again shows that time-constant matching between cyber and physical circuit elements gives a systematic design approach.

IV. SIMULATION RESULTS

To substantiate the performance of the above-mentioned PI and PR compensators with control gains obtained based on the
circuit-equivalent model, two sets of simulations configured with system parameters listed in Table I were carried out. Figures 4 and 5 present the time-domain simulation results for the PI compensator and PR compensator, respectively. Due to the symmetry of the two-dimensional components, we only show the $d$-axis component $i_d$ in the $dq$ reference-frame and the $\alpha$-axis component $i_\alpha$ in the $\alpha\beta$ reference-frame.

In Fig. 4, we can observe that the dc current tracking performance ($i_d \rightarrow i_d^*$) follows a first-order system response as required. In Fig. 5, we can also see that the current $i_\alpha$ tracks the current reference $i_\alpha^*$. This is accomplished with zero steady-state amplitude and instantaneous phase-tracking error and desired transient dynamics are achieved.

V. CONCLUSIONS & FUTURE WORK

In this paper, we derived unified circuit-equivalent representations capturing controller and filter dynamics for current-controlled inverters. Results were given in both the synchronous $dq$ and stationary $\alpha\beta$ reference frames. A variety of circuit-theoretic notions and rules of thumb were then leveraged for controller design. As part of future work, we will aim to incorporate the PLL and dc-link dynamics into the circuit-based representation.

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