Reduced-order Aggregate Model for Parallel-connected Single-phase Inverters

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Abstract—This paper outlines a reduced-order aggregate dynamical model for parallel-connected single-phase grid-connected inverters. For each inverter, we place no restrictions on the converter topology and merely assume that the ac-side switch-averaged voltage can be controlled via pulsewidth modulation. The ac output of each inverter interfaces through an \textit{LCL} filter to the grid. The closed-loop system contains a phase locked loop for grid synchronization, and real- and reactive-power control are realized with inner and outer PI current- and power-control loops. We derive a necessary and sufficient set of parametric relationships to ensure that a reduced-order aggregated state-space model for an arbitrary number of such paralleled inverters has the same model order and structure as any single inverter. We also present reduced-order models for the settings where the real- and reactive-power setpoints are different and where the inverters have different power ratings. We anticipate the proposed model being useful in analyzing the dynamics of large collections of parallel-connected inverters with minimal computational complexity. The aggregate model is validated against measurements obtained from a multi-inverter experimental setup consisting of three 750-VA paralleled grid-connected inverters, hence establishing robustness of the analytical result to parametric variations seen in practice.

Index Terms—Model reduction, phase-locked loop, single-phase inverter, voltage-source inverter.

I. INTRODUCTION

RAPID adoption of renewable sources of generation (e.g., photovoltaic (PV) energy conversion systems) and flexible loads (e.g., electric vehicles) has increased the number of power electronics inverters installed on the ac power grid. Scalable models that present limited computational burden will be critical to model and analyze the collective dynamics of large numbers of inverters in next-generation power networks [1]. To motivate the need for modeling strategies that can be applied to complex inverter systems, consider the relatively small island of Oahu which already has over 800,000 microinverters [2]. This number is expected to grow significantly as Hawaii aims to meet the goal of obtaining 100\% of its energy from renewable sources [3]. Although the Hawaiian system is at the forefront of renewable adoption, it presents a glimpse at anticipated worldwide trends [4].

The disparity in ratings between individual inverters (no larger than a few MVA) and synchronous generators (several hundred MVA) implies that if the same net load were served with power electronics instead of generators, there would be an orders-of-magnitude increase in the number of energy-conversion interfaces (from a few thousand generators to potentially millions of inverters across a large-scale synchronous grid). Evaluating the stability and resilience of future power networks will therefore require accurate dynamical models for large collections of inverters that present limited computational burden. However, development of such models is challenged by the complexity of inverter dynamics (for instance, the particular model we examine in this paper is nonlinear, and composed of 16 states) and the sheer number of inverters that will eventually be commonplace on the ac power grid. To address the challenge of model complexity in multi-inverter systems, we propose an aggregate reduced-order state-space model for an arbitrary number of single-phase grid-tied inverters connected in parallel. While our analytical result is presented for identical inverters, we experimentally validate our findings which immediately establishes robustness to parametric variations that are likely to be seen in practice. We also present extensions of the main result on model reduction to cover cases when the power setpoints of the inverters are all different and the power ratings of the inverters are all different.

We examine the ac-timescale dynamics of a single-phase voltage source inverter (VSI) with an output \textit{LCL} filter. To ensure broad applicability across VSI topologies, we only assume that the switch-averaged voltage across the ac terminals is controllable via pulse width modulation and we neglect switch-level dynamics. The control architecture is composed of an inner current-control loop, an outer power-control loop, and a phase locked loop (PLL) for grid synchronization. This filter and control architecture are prototypical and it ensures broad applicability of the results. The state-space model that captures the

dynamics of the inverter is composed of 16 states. The contributions of the paper are threefold:

1) For a parallel collection of inverters, we derive a necessary and sufficient set of parametric relationships for an aggregate reduced-order inverter model to have a state-space model with the same dimension and structure as any individual inverter. This implies that the parallel system is described by a 16-order dynamical model.

2) We derive parameters for an aggregate reduced-order model (with the same order and structure as any individual inverter) for the case where the real and reactive-power setpoints for the inverters are all different.

3) We derive parameters for an aggregate reduced-order model (with the same order and structure as any individual inverter) for the case where the power ratings of the inverters are all different.

In general, the identical state-space model structure implies that from a topological vantage point, the aggregated equivalent model also maps to an inverter with an identical state-space model structure as any individual inverter, i.e., it is described by a 16-order dynamical model, and \( A^*, B^*, \) and \( g^*(\cdot, \cdot) \) have the same form as \( A, B, \) and \( g(\cdot, \cdot) \). With regard to inverter dynamics, most of the related literature has predominantly focused on reduced-order models for individual grid-connected inverters, where the parallel aggregation approach discussed here.

Given the landscape of related work discussed above, this work addresses a key gap in the literature pertaining to the dynamics of grid-connected multi-inverter systems. This paper significantly builds upon and extends our preliminary work in [26] where we developed similar aggregated models for parallel-connected three-phase inverters. Here, we examine the (admittedly different) filter and controller dynamics for single-phase inverters which will conceivably be more dominant in number in future distribution networks. As a further and important contribution, we provide experimental validation of our approach with a multi-inverter setup composed of three 750 VA grid-tied single-phase inverters. We note that the parallel aggregation approach proposed here provides the base for a broader set of aggregation techniques that can account for the network connection of the inverters. For instance, in [27], three-phase inverters are transferred to an auxiliary bus with the aid of auxiliary transformers, and subsequently aggregated using the parallel aggregation approach discussed here.

The remainder of this manuscript is organized as follows: In Section II, we establish mathematical notation and describe the grid-tied single-phase inverter model. The reduced-order model for a collection of these inverters connected in parallel is derived in Section III. We validate the model-reduction method by comparing numerical simulation results with results from the experimental prototype in Section IV. Finally, concluding remarks and directions for future work are in Section V.

II. PRELIMINARIES AND INVERTER DYNAMICAL MODEL

In this section, we first introduce mathematical notation used in the manuscript. Then we describe the single-phase inverter model, and develop a standard state-space model representation.
(x^d, x^a) with the following rotation matrix [29]:

\[
\begin{bmatrix}
  x^d \\
  x^a
\end{bmatrix} =
\begin{bmatrix}
  \cos \delta & \sin \delta \\
  -\sin \delta & \cos \delta
\end{bmatrix}
\begin{bmatrix}
  x^\alpha \\
  x^\beta
\end{bmatrix},
\]

where $\delta$ is the instantaneous angle generated by the PLL. As seen in Fig. 2, the PLL is in feedback with the dq transformation. The role of the PLL is to modulate the value of the PLL angle, $\delta$, such that the $d$-axis component of the grid voltage, $v_{g}^d$, is driven asymptotically to zero. From the definition of the $\alpha$- and $\beta$-components of $v_{g}$ and the dq transformation in (2), it can be shown that if $v_{g}^d = 0$, then $\delta$ is the instantaneous phase angle of $v_{g}$. (See Appendix A for a short derivation.)

Controller and filter dynamics: The internal controllers in the PLL comprise a low-pass filter with cut off frequency $\omega_{c,PLL}$ and a PI controller with proportional and integral gains given by $k_{PLL}^p$ and $k_{PLL}^i$, respectively. The PLL dynamics are given by

\[
\begin{align}
\frac{dv_{PLL}^d}{dt} &= \omega_{c,PLL}(v_{g}^d - v_{PLL}^d), \quad \text{(3a)} \\
\frac{dv_{PLL}^\alpha}{dt} &= -v_{PLL}^\alpha, \quad \text{(3b)} \\
\frac{d\delta}{dt} &= \omega_{nom} - k_{PLL}^p v_{PLL}^d + k_{PLL}^i \phi_{PLL} =: \omega_{PLL}, \quad \text{(3c)} \\
\frac{dv_{g}^\beta}{dt} &= \omega_{PLL}(v_{g}^d - v_{g}^\beta) - \frac{d}{dt} v_{g}^\alpha, \quad \text{(3d)}
\end{align}
\]

where $\omega_{nom}$ is the nominal grid frequency (e.g., $2\pi \times 60$ or $2\pi \times 50$ rad/s). We apply (1) to $v_{g}$ to obtain the dynamics of $v_{g}^\beta$ in (3d), and we apply (2) to $v_{g}$ and $v_{g}^\beta$ to obtain $v_{g}^d$ which feeds into (3a). From above, we can see that $v_{g}^d = v_{PLL}^d = 0$ in steady-state. Furthermore, when the grid frequency is $\omega_{nom}$, it follows that $\delta = \omega_{PLL} = \omega_{nom}$. Note that we assume the first derivative of $v_{g}$, i.e., $\frac{dv_{g}^d}{dt}$, is well defined.

The LCL filter is composed of inverter-side inductance $L_i$, grid-side inductance, $L_g$, and filter capacitance, $C_f$. The dynamics introduced by the LCL filter in the $\alpha\beta$ frame are given by

\[
\begin{align}
\frac{d}{dt} i_{i}^\alpha &= \frac{1}{L_i} (-R_i i_{i}^\alpha + v_{g}^\alpha - v_{l}^\alpha), \quad \text{(4a)} \\
\frac{d}{dt} i_{i}^\beta &= \omega_{PLL}(i_{i}^\alpha - i_{i}^\beta) - \frac{d}{dt} i_{l}^\alpha, \quad \text{(4b)}
\end{align}
\]
\[ \frac{d}{dt} i^\alpha_g = \frac{1}{L_g} (-R_g i^\alpha_g + v_i^\alpha - v_g), \quad (4c) \]
\[ \frac{d}{dt} i^\beta_g = \omega_{PLL} (i^\alpha_g - i^\beta_g) - \frac{d}{dt} i^\beta_g, \quad (4d) \]
\[ \frac{d}{dt} v_i^\alpha = R_L \left( \frac{d}{dt} i^\alpha - d_{\text{avg}} i^\beta \right) + \frac{1}{C} (i^\alpha - i^\alpha_g), \quad (4e) \]
\[ \frac{d}{dt} v_i^\beta = \omega_{PLL} (v_i^\alpha - v_i^\beta) - \frac{d}{dt} v_i^\beta, \quad (4f) \]

where the \( \alpha \)-component dynamics are derived from fundamental circuit laws, and the \( \beta \)-component expressions result from the application of (1) to the corresponding \( \alpha \)-component dynamics.

The power controller (PC) consists of two PI controllers with gains \( k_{PC}^P \) and \( k_{PC}^Q \) and two low-pass filters with cut-off frequency \( \omega_{c,PC} \) for the \( d \) and \( q \) components. The real- and reactive-power setpoints, \( p^* \) and \( q^* \), act as inputs to the power controller and its outputs are current references for the downstream current controller. These are generated as follows:

\[ i_{\text{avg}}^\alpha = k_{PC}^P (q^* - q_{\text{avg}}) + k_{PC}^Q \int (q^* - q_{\text{avg}}), \quad (5a) \]
\[ i_{\text{avg}}^\beta = k_{PC}^P (p^* - p_{\text{avg}}) + k_{PC}^Q \int (p^* - p_{\text{avg}}), \quad (5b) \]

where \( p_{\text{avg}} \) and \( q_{\text{avg}} \) are the outputs of the low-pass filters, with inputs to be the inverter real- and reactive-power outputs measured at the grid terminals, \( p \) and \( q \), respectively. In particular, with reference to Fig. 2 and with the aid of elementary trigonometric operations we have

\[ p = \frac{1}{2} (v_g^p g + v_g^q g), \quad q = \frac{1}{2} (v_g^p g - v_g^q g), \quad (6) \]

and as discussed above,

\[ \frac{d}{dt} p_{\text{avg}} = \omega_{c,PC} (p - p_{\text{avg}}), \quad \frac{d}{dt} q_{\text{avg}} = \omega_{c,PC} (q - q_{\text{avg}}). \quad (7) \]

The real-power setpoint, \( p^* \), reflects the real power that is ultimately generated by an upstream input-stage controller and dc-link voltage controller acting in concert. For instance, for a PV application, \( p^* \) could be approximated from scaled irradiance data assuming accurate and fast maximum power point tracking. Similarly, the reactive power setpoint, \( q^* \), either be fixed at zero to reflect unity power factor operation or may alternatively be generated by a Volt/VAR controller. For the sake of generality, we will simply consider \( p^* \) and \( q^* \) as generic power controller inputs for the remainder of the paper.

The current controller (CC) is composed of two PI controllers with gains \( k_{CC}^P \) and \( k_{CC}^Q \). and as outputs, it generates the voltage references for the PWM modulation block:

\[ v_i^\alpha = v_i^\alpha + k_{CC}^P (i_{\text{avg}}^\alpha - i^\alpha) + k_{CC}^Q \int (i_{\text{avg}}^\alpha - i^\alpha) \quad (8a) \]
\[ v_i^\beta = v_i^\beta + k_{CC}^P (i_{\text{avg}}^\beta - i^\beta) + k_{CC}^Q \int (i_{\text{avg}}^\beta - i^\beta) \quad (8b) \]

The addition of the feedforward terms \( v_i^\alpha \) and \( v_i^\beta \) (obtained by applying (2) to \( v_i^\alpha \) and \( v_i^\beta \)) is standard practice, and intended to improve dynamic performance [30]. Suppose the VSI is ideal (see Fig. 2), then the terminal inverter voltage is given by:

\[ v_i^* \approx v_i^\alpha = v_i^\alpha \cos \delta - v_i^\beta \sin \delta, \quad (9) \]

where \( \delta \) is the instantaneous PLL angle. This approximation implies that the inverter terminal voltage follows the commanded reference perfectly and without delay.

### C. State-space Representation of Inverter Dynamics

The dynamics of the \( LCL \) filter, PLL, power controller, and current controller for an individual inverter are now expressed in state-space form to facilitate analysis. To this end, corresponding to the power and current controllers, we will find it useful to introduce the auxiliary dynamics

\[ \frac{d}{dt} \phi^p = p^* - p_{\text{avg}}, \quad \frac{d}{dt} \phi^q = q^* - q_{\text{avg}} \quad (10) \]

\[ \frac{d}{dt} \gamma^d = i_{\text{avg}}^\alpha - i^\alpha, \quad \frac{d}{dt} \gamma^q = i_{\text{avg}}^\beta - i^\beta \quad (11) \]

With these definitions in place, the dynamics (3a)–(11) can be represented in a compact state-space form

\[ \dot{x} = Ax + Bu_1 + B_2 u_2 + g(x, u_1, u_2) \quad (12) \]

where the state vector, \( x \), and inputs \( u_1 \), \( u_2 \) are given by

\[ x = [v_i^\alpha, \tilde{i}_g^\alpha, v_i^\beta, \tilde{i}_g^\beta, v_{PLL}, \phi_{PLL}, \phi_L, \phi_T]^T \quad (13) \]

\[ u_1 = [p^*, q^*]^T, \quad u_2 = [v_i^\alpha, \tilde{i}_g^\alpha]^T \quad (14) \]

In order to show the entries of matrices \( A \in \mathbb{R}^{16 \times 16} \), \( B_1 \in \mathbb{R}^{16 \times 2} \), and \( B_2 \in \mathbb{R}^{16 \times 2} \), let us partition the state vector as \( x = [x_{LCL}, x_{CC}, x_{PC}, x_{PLL}]^T \), where \( x_{LCL} = [v_i^\alpha, \tilde{i}_g^\alpha, v_i^\beta, \tilde{i}_g^\beta, v_{PLL}, \phi_{PLL}, \phi_L, \phi_T]^T \), \( x_{CC} = [\gamma^d, \gamma^q]^T \), \( x_{PC} = [p_{\text{avg}}, q_{\text{avg}}, \phi^p, \phi^q]^T \), and \( x_{PLL} = [v_i^\beta, v_{PLL}, \phi_{PLL}, \delta]^T \). Then, we can write (12) as

\[
\begin{bmatrix}
\dot{x}_{LCL} \\
\dot{x}_{CC} \\
\dot{x}_{PC} \\
\dot{x}_{PLL}
\end{bmatrix} = 
\begin{bmatrix}
A_{LCL} & 0_{6 \times 2} & 0_{6 \times 4} & 0_{6 \times 4} \\
0_{2 \times 6} & 0_{2 \times 2} & A_{CC} & 0_{2 \times 4} \\
0_{4 \times 6} & 0_{4 \times 2} & A_{PC} & 0_{4 \times 4} \\
0_{4 \times 6} & 0_{4 \times 2} & 0_{4 \times 4} & A_{PLL}
\end{bmatrix}
\begin{bmatrix}
x_{LCL} \\
x_{CC} \\
x_{PC} \\
x_{PLL}
\end{bmatrix}
+ 
\begin{bmatrix}
0_{6 \times 2} \\
B_{CC} \\
B_{PC} \\
0_{4 \times 2}
\end{bmatrix} u_1 + 
\begin{bmatrix}
B_{LCL} \\
0_{2 \times 2} \\
0_{4 \times 2} \\
B_{PLL}
\end{bmatrix} u_2 + g(x, u_1, u_2),
\]

where entries of the nonzero sub-matrices \( A_{LCL}, A_{CC}, A_{PC}, A_{PLL}, B_{CC}, B_{PC}, B_{LCL}, B_{PLL} \), and the function \( g(x, u_1, u_2) : \mathbb{R}^{16} \times \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^{16} \) are spelled out in Appendix B.
III. AGGREGATION OF PARALLEL-CONNECTED INVERTERS

In this section, we first introduce parametric scalings required to realize the aggregate model for the parallel-connected inverters. Next, we prove that the aggregate model indeed captures all the scalings in pertinent states (currents, voltages, internal-control states) for the uniform setting as well as in cases with heterogeneous power setpoints and ratings.

A. Parametric Scalings and Structure of Aggregate Model

We consider $N$ identical single-phase inverters (with model described in Section II) that have the same setpoints, $p^{*}$ and $q^{*}$, connected in parallel to a grid bus. We are interested in an aggregated reduced-order model with the same structure and dimension as the model in (12):

$$\dot{x}^r = A^r x^r + B^r_1 u^r_1 + B^r_2 u^r_2 + g^r(x^r, u^r_1, u^r_2).$$ (16)

In particular, we desire matrices $A^r \in \mathbb{R}^{16 \times 16}$, $B^r_1 \in \mathbb{R}^{16 \times 2}$, $B^r_2 \in \mathbb{R}^{16 \times 2}$, and function $g^r: \mathbb{R}^{16 \times 2} \times \mathbb{R}^2 \rightarrow \mathbb{R}^{16}$ have the same structure and dimension as $A, B_1, B_2,$ and $g$ in (12) (implying that the control architecture and output-filter arrangement of the aggregated model are the same as that in an individual inverter); entries of state vector $x^r$ and inputs $u^r_1, u^r_2$ to have the same notation as states in $x$ and inputs $u_1, u_2$. In our main result, we demonstrate that if and only if the following relationships hold between parameters of the reduced-order and original models:

$$C^r_1 = NC_1, \quad R^r_1 = \frac{R_1}{N}, \quad L^r_1 = \frac{L_1}{N}, \quad R^r_g = \frac{R_g}{N},$$ (17a)

$$\frac{R^r_1}{L^r_1} = \frac{R_1}{L_1} \frac{k^{p\text{PLL}}_{\text{avg}}}{k^{p\text{PLL}}_{\text{avg}}} = \frac{k^{p\text{PLL}}_{\text{avg}}}{k^{p\text{PLL}}_{\text{avg}}} = \frac{k^{p\text{PLL}}_{\text{avg}}}{k^{p\text{PLL}}_{\text{avg}}},$$ (17b)

and the reference power settings of the lumped-parameter aggregate inverter model in Fig. 3a are $N$ times those of the inverter model in Fig. 2, the current- and power-related states $i_{q1}^{\alpha}, i_{q1}^{\beta}, i_{q1}^{\gamma}, i_{q1}^{\delta}, \gamma_{q1}, \gamma_{q2}, p_{\text{avg}}, q_{\text{avg}}, \phi_{\text{avg}}, \phi_{\text{avg}}$, and the voltage-related states in the reduced-order model are $N$ times the corresponding ones in an individual inverter. Furthermore, the voltage- and PLL-related states in the reduced-order model $v_{q1}^{\alpha}, v_{q1}^{\beta}, v_{q1}^{\gamma}, v_{q1}^{\delta}, v_{\text{PLL}}, \phi_{\text{PLL}}, \delta$ are the same as those in any inverter in the parallel combination. This is consistent with the electrical behavior of a parallel connection of current (or power) sources.

In establishing the above result, we will have established that the reduced-order aggregate model in Fig. 3a captures the dynamics of the parallel collection. Put differently, we will mathematically establish the equivalence illustrated in Fig. 3b. It is worth emphasizing that the dynamical model of an individual inverter has 16 states, and so modeling the dynamics of every inverter in an $N$-inverter parallel collection would require a $16N$-order state-space model. By contrast, the reduced-order model has the same structure as any individual inverter, and is hence described only by 16 states.

B. Main Result: Validating the Aggregate Model

We now state and prove the main result of this paper.

**Theorem 1. (Aggregation of parallel-connected identical single-phase inverters):** Consider the dynamical model for the single-phase inverter specified in (12). Permute $x$ in (13) as

$$\vec{x} = [i_1^{\alpha}, i_1^{\beta}, i_1^{\gamma}, i_1^{\delta}, \gamma_1, \gamma_2, p_{\text{avg}}, q_{\text{avg}}, \phi_{\text{avg}}, \phi_{\text{avg}}, v_1^{\alpha}, v_1^{\beta}, v_1^{\gamma}, v_1^{\delta}, v_{\text{PLL}}, \phi_{\text{PLL}}, \delta]^T,$$ (18)

and also permute $x^r$ (corresponding to the reduced-order model (16)) the same way, denoting the permuted vector by $\vec{x}^r$. Denote $\vec{x}(t)$ to be the solution to the permuted version of (12) with initial condition $\vec{x}(t_0)$ and inputs $u_1, u_2$; and $\vec{x}^r(t)$ to be the solution to the permuted version of (16) with initial condition $\vec{x}^r(t_0)$ and inputs $u_1^r, u_2^r$. Suppose initial conditions are such that $\vec{x}^r(t_0) = \text{diag}(\Psi)\vec{x}(t_0)$, where the scaling vector, $\Psi := [N[1]_T^T, 1]_T^T$, and the inputs: $u_1^r = N u_1, u_2^r = u_2$ (see (14)). The states of the reduced-order model and the individual inverter are related as:

$$\vec{x}^r(t) = \text{diag}(\Psi)\vec{x}(t), \quad \forall t \geq t_0,$$ (19)

if and only if their parameters are related as in (17a)–(17b).

In particular, given the definition of the scaling vector, $\Psi$, (19) establishes the following relationships between states of the reduced-order model and those in the individual inverter model.
\begin{align}
\forall t \geq t_0: & \quad [t_1^{\alpha,r}, t_1^{\beta,r}, t_1^g, t_0^{\gamma,n}, t_{av}, v_{avg}, \phi_{avg}, \phi_{avg}, \phi_{avg}]^T \\
& = N[t_1^{\alpha,r}, t_1^{\beta,r}, t_1^g, t_0^{\gamma,n}, t_{av}, v_{avg}, \phi_{avg}, \phi_{avg}, \phi_{avg}]^T, \quad (20a) \\
& = [v_1^\gamma, v_1^\beta, v_1^g, v_{PLL}, \phi_{PLL}, \delta]^T. \quad (20b)
\end{align}

**Proof:** Let us define \( \tilde{x} = \tilde{x} - \text{diag}(\Psi) \tilde{x} \). The dynamics of \( z \) are given by
\[ z = \tilde{x} - \text{diag}(\Psi) \tilde{x} = \tilde{A} \tilde{x} + \tilde{B}_1 u_1 + \tilde{B}_2 u_2 + g'(\tilde{x}, u_1, u_2) - \text{diag}(\Psi) \tilde{A} \tilde{x} - \text{diag}(\Psi) \tilde{B}_1 u_1 - \text{diag}(\Psi) \tilde{B}_2 u_2 - \text{diag}(\Psi) \tilde{g}(\tilde{x}, u_1, u_2), \quad (21) \]
where matrices \( \tilde{A}, \tilde{B}_1, \tilde{B}_2, \tilde{A}', \tilde{B}_1', \tilde{B}_2' \) and functions \( \tilde{g}(\tilde{x}, u_1, u_2), \tilde{g}'(\tilde{x}, u_1, u_2) \) are appropriately permuted versions of corresponding matrices and functions in (12) and (16). We will now show that \( z = 0, \forall t \geq t_0 \) when \( z(t_0) = \tilde{x} - \text{diag}(\Psi) \tilde{x}(t_0) = 0 \). This would further imply that \( z(t) = \tilde{x} - \text{diag}(\Psi) \tilde{x}(t) = 0, \forall t \geq t_0 \), as claimed in the statement of the Theorem.

Partition \( \tilde{x} = [\tilde{x}_1^T, \tilde{x}_2^T]^T \), where \( \tilde{x}_1 = [\tilde{v}_1^\alpha, \tilde{v}_1^\beta, \tilde{v}_1^g, \tilde{\gamma}_1^g, \tilde{\gamma}_1^n, \tilde{p}_{avg}, \tilde{v}_{avg}, \tilde{\phi}_{avg}, \tilde{\phi}_{avg}]^T \) and \( \tilde{x}_2 = [v_1^\gamma, v_1^\beta, v_1^g, v_{PLL}, \phi_{PLL}, \delta]^T \), and also partition \( \tilde{x}' \) the same way. Then, we partition the appropriately permuted versions of (12) and (16) as
\begin{align}
& \begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2
\end{bmatrix} = \begin{bmatrix}
\tilde{A}_{11} & \tilde{A}_{12} \\
\tilde{A}_{21} & \tilde{A}_{22}
\end{bmatrix} \begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2
\end{bmatrix} + \begin{bmatrix}
\tilde{B}_1 \\
\tilde{B}_2
\end{bmatrix} u_1 + \begin{bmatrix}
\tilde{B}_1 \\
\tilde{B}_2
\end{bmatrix} u_2 \\
& + g(\tilde{x}, u_1, u_2), \quad (22)
\end{align}
\begin{align}
& \begin{bmatrix}
\tilde{x}_1' \\
\tilde{x}_2'
\end{bmatrix} = \begin{bmatrix}
\tilde{A}'_{11} & \tilde{A}'_{12} \\
\tilde{A}'_{21} & \tilde{A}'_{22}
\end{bmatrix} \begin{bmatrix}
\tilde{x}_1' \\
\tilde{x}_2'
\end{bmatrix} + \begin{bmatrix}
\tilde{B}_1' \\
\tilde{B}_2'
\end{bmatrix} u_1 + \begin{bmatrix}
\tilde{B}_1' \\
\tilde{B}_2'
\end{bmatrix} u_2 \\
& + \tilde{g}'(\tilde{x}', u_1, u_2), \quad (23)
\end{align}

From the definition of matrices \( A_{LCL}, A_{ACC}, A_{PC}, A_{PLL}, B_{ACC}, B_{PC}, B_{LCL}, B_{PLL} \) in Appendix B, and the parametric scalings established in (17a)–(17b), we note the following:
\begin{align}
& \tilde{A}_{11} = \tilde{A}_{11}, \quad \tilde{A}_{12} = N \tilde{A}_{21}, \quad \tilde{A}_{21} = \frac{1}{N} \tilde{A}_{21}, \quad \tilde{A}_{22} = \tilde{A}_{22}, \\
& \tilde{B}_1' = \tilde{B}_1, \quad \tilde{B}_2' = \frac{1}{N} \tilde{B}_2, \quad \tilde{B}_1 = N \tilde{B}_2, \quad \tilde{B}_2 = \tilde{B}_2. \quad (24)
\end{align}
Then, we have
\begin{align}
& \text{diag}(\Psi) \tilde{A} = \begin{bmatrix}
N \tilde{A}_{11} & N \tilde{A}_{12} \\
N \tilde{A}_{21} & N \tilde{A}_{22}
\end{bmatrix} = \begin{bmatrix}
N \tilde{A}_{11} & N \tilde{A}_{12} \\
N \tilde{A}_{21} & N \tilde{A}_{22}
\end{bmatrix} = \tilde{A}' \text{diag}(\Psi), \quad (25) \\
& \text{diag}(\Psi) \tilde{B}_1 = \begin{bmatrix}
N \tilde{B}_1 \\
N \tilde{B}_2
\end{bmatrix} = \begin{bmatrix}
N \tilde{B}_1 \\
N \tilde{B}_2
\end{bmatrix} = \tilde{B}_1', \quad (26) \\
& \text{diag}(\Psi) \tilde{B}_2 = \begin{bmatrix}
N \tilde{B}_2 \\
N \tilde{B}_2
\end{bmatrix} = \begin{bmatrix}
\tilde{B}_2 \\
\tilde{B}_2
\end{bmatrix} = \tilde{B}_2'. \quad (27)
\end{align}

The next step is to show that \( \tilde{g}'(\text{diag}(\Psi) \tilde{x}, u_1, u_2) = \text{diag}(\Psi) \tilde{g}'(\tilde{x}, u_1, u_2). \) Let \( \tilde{g}'(\tilde{x}) \) and \( \tilde{g}'(\text{diag}(\Psi) \tilde{x}) \) denote the \( t \)-th entry of \( \tilde{g}(\tilde{x}, u_1, u_2) \) and \( \text{diag}(\Psi) \tilde{g}(\tilde{x}, u_1, u_2) \), respectively. The nonzero entries of \( \tilde{g}'(\text{diag}(\Psi) \tilde{x}, u_1, u_2) \) are related to the corresponding entries of \( \tilde{g}(\tilde{x}, u_1, u_2) \) through:
\[ \tilde{g}'(\text{diag}(\Psi) \tilde{x}, u_1, u_2) = \tilde{g}(\tilde{x}, u_1, u_2). \]

Notice that the PLL dynamics (3a)–(3d) are decoupled from the remainder of the states in the state-space model. Therefore, the parameters of the PLL in the individual and reduced-order models are the same, and we can conclude that:
\[ v^g = v^g, \quad v^g_{PLL} = v^g_{PLL}, \quad \phi_{PLL} = \phi_{PLL}, \quad \delta' = \delta. \]

Now, consider the function \( h(x', u_1, u_2) : \mathbb{R}^6 \times \mathbb{R}^6 \times \mathbb{R}^2 \rightarrow \mathbb{R}^6 \), which is defined to have the same structure as \( \tilde{g}'(\tilde{x}', u_1, u_2) \) except that its 13th, 14th, and 16th entries are 0. (See Appendix B for details.) Then, the following holds:
\[ \tilde{g}'(\tilde{x}', u_1, u_2) - \tilde{g}(\text{diag}(\Psi) \tilde{x}', u_1, u_2) = h(\tilde{x}' - \text{diag}(\Psi) \tilde{x}', 0, 0, 0). \]

Using identities (25)–(30) in (21), we have
\[ z = \tilde{A} \tilde{x} - \text{diag}(\Psi) \tilde{x} + h((\tilde{x}' - \text{diag}(\Psi) \tilde{x}), 0, 0, 0) = \tilde{A} z + h(z, 0, 0, 0). \]
For the other direction, given that \( z(t) = \tilde{z} - \text{diag}(\Psi) \tilde{x}, \forall t \geq t_0, u_1 = N u_1, \) and \( u_2 = u_2, (21) \) can be written as
\[
0_{15} = (\tilde{A} \text{diag}(\Psi) - \text{diag}(\Psi) \tilde{A}) \tilde{x} + (N \tilde{B}_1 - \text{diag}(\Psi) \tilde{B}_1) u_1 \\
+ (\tilde{B}_2 - \text{diag}(\Psi) \tilde{B}_2) u_2 + \tilde{g}'(\text{diag}(\Psi) \tilde{x}, N u_1, u_2) \\
- \text{diag}(\Psi) \tilde{g}(\tilde{x}, u_1, u_2).
\]
(32)
The equality above is satisfied when the following identities hold:
\[
\tilde{A} \text{diag}(\Psi) = \text{diag}(\Psi) \tilde{A}, \quad N \tilde{B}_1 = \text{diag}(\Psi) \tilde{B}_1,
\]
\[
\tilde{B}_2 = \text{diag}(\Psi) \tilde{B}_2,
\]
\[
\tilde{g}'(\text{diag}(\Psi) \tilde{x}, N u_1, u_2) = \text{diag}(\Psi) \tilde{g}(\tilde{x}, u_1, u_2).
\]
(33)
(34)
It emerges that \( R_i, L_i, k_{CC}^i, \) and \( k_{CC}^i \) always appear in the identities above as fractions: \( \frac{R_i}{L_i}, k_{CC}^i, \) and \( k_{CC}^i \). Therefore, these parameters relate to those in the reduced-order model through (17b). The remainder of the parameters can be determined straightforwardly from (33) and (34); they are given uniquely by (17a), including the unchanged parameters (i.e., those which are not mentioned in (17a) and (17b)). This concludes the proof.

C. Corollaries for Heterogeneous Settings

We now present two corollaries. In the first, we examine inverters with different reference-power setpoints for both active and reactive power. In this particular case, the relationships between the states of the reduced-order model and those of the individual inverters \( \ell = 1, \ldots, N \) are as follows \( \forall t \geq t_0 \):
\[
\[v^r_{g,i}, v^q_{g,i}, v^r_{PLL,i}, v^q_{PLL,i}, f^\alpha_{PLL,i}, f^\beta_{PLL,i}, f^\gamma_{PLL,i}, f^\delta_{PLL,i} \] \text{T} = N \sum_{i=1}^{N} \[v^\alpha_{g,i}, v^\beta_{g,i}, v^\gamma_{g,i}, v^\delta_{g,i}, f^\alpha_{PLL,i}, f^\beta_{PLL,i}, f^\gamma_{PLL,i}, f^\delta_{PLL,i} \] \text{T}.
\]
(35)
This model is useful in, e.g., PV systems where the incident irradiation might be different for different inverters (hence resulting in different values for \( \rho_p \)) and where local-voltage control may be implemented by modulating reactive-power injections (hence resulting in different values of \( q^* \)).

In the second corollary, we examine inverters with different power ratings, and derive an aggregate model with currents that scale systematically. To formalize this, we define a power-scaling parameter \( \kappa_\ell \) for the \( \ell \)-th inverter as [26]:
\[
\kappa_\ell = \frac{p_{\text{rated},\ell}}{p_{\text{phase}}},
\]
(36)
where \( p_{\text{rated},\ell} \) and \( p_{\text{phase}} \) denote the rated power of the \( \ell \)-th inverter in the parallel system and system-wide base value, respectively. Without loss of generality, we assume that the inverter model in Fig. 2 has a power rating equal to the base value.

We also introduce the notion of an equivalent power-scaling parameter:
\[
\kappa := \sum_{\ell=1}^{N} \kappa_\ell.
\]
(37)
The states of the reduced-order model relate to those in the individual inverters \( \ell = 1, \ldots, N \) and the unscaled inverter (i.e., inverter model with rating equal to the base value) as follows \( \forall t \geq t_0 \):
\[
\[v^r_{g,\ell}, v^q_{g,\ell}, v^r_{PLL,\ell}, v^q_{PLL,\ell}, f^\alpha_{PLL,\ell}, f^\beta_{PLL,\ell}, f^\gamma_{PLL,\ell}, f^\delta_{PLL,\ell} \] \text{T} = \sum_{\ell=1}^{N} \[v^\alpha_{g,\ell}, v^\beta_{g,\ell}, v^\gamma_{g,\ell}, v^\delta_{g,\ell}, f^\alpha_{PLL,\ell}, f^\beta_{PLL,\ell}, f^\gamma_{PLL,\ell}, f^\delta_{PLL,\ell} \] \text{T}.
\]
(38)
Formal results establishing these two aspects follow next.

**Corollary 1.** (Aggregation of parallel-connected identical single-phase inverters with different reference-power setpoints): Let us denote \( x_i, p^r_i, \) and \( q^r_i \) as the state vector, real-, and reactive-power setpoints of the \( \ell \)-th inverter in the parallel system. Permute \( x_i \) the same way as in (18), denoting the permuted vector as \( \tilde{x}_\ell \). Partition \( \tilde{x}_\ell = \left[ \lambda_i, v^r_{\ell}, v^q_{\ell}, x_{PLL,\ell} \right] \text{T} \), where \( \lambda_\ell = \left[ \left[ v^\alpha_{\ell}, v^\beta_{\ell}, v^\gamma_{\ell}, v^\delta_{\ell}, f^\alpha_{PLL,\ell}, f^\beta_{PLL,\ell}, f^\gamma_{PLL,\ell}, f^\delta_{PLL,\ell} \right] \text{T}, \quad v^\alpha_{\ell} = \left[ v^\alpha_{g,\ell}, v^\alpha_{PLL,\ell}, v^\alpha_{PLL,\ell}, v^\alpha_{PLL,\ell} \right] \text{T}, \text{ and } x_{PLL,\ell} = \left[ v^\beta_{PLL,\ell}, v^\gamma_{PLL,\ell}, v^\delta_{PLL,\ell} \right] \text{T} \). The same power-scalable parameters relate to different inverters through (17a)–(17b).

**Proof:** The proof is provided in Appendix C.

**Corollary 2.** (Aggregation of parallel-connected single-phase inverters with heterogeneous power ratings): The parameters of each inverter are related to the unscaled inverter through
\[
C_{\ell,\ell} = \kappa_\ell C_1, \quad R_{\ell,\ell} = \frac{R_1}{\kappa_\ell}, \quad L_{g,\ell} = \frac{L_g}{\kappa_\ell}, \quad R_{g,\ell} = \frac{R_g}{\kappa_\ell},
\]
(41a)
\[
\frac{R_{\ell,\ell} L_{\lambda,\ell}}{L_{\lambda,\ell}} = \frac{R_1 L_1}{L_1}, \quad \frac{k_{CC}^p_{\ell}}{L_{\lambda,\ell}} = \frac{k_{CC}^p}{L_1}, \quad \frac{k_{CC}^q_{\ell}}{L_{\lambda,\ell}} = \frac{k_{CC}^q}{L_1}.
\]
(41b)
and parameters not mentioned are unchanged. Suppose the reference-power setpoints for each inverter are $p_i^* = \kappa_i p^*$ and $q_i^* = \kappa_i q^*$. The parameters of the reduced-order model are related to the unscaled inverter through

$$
\begin{align*}
C_i^* &= \kappa C_i, \quad R_i^* = \frac{R_i}{\kappa}, \quad L_i^* = \frac{L_i}{\kappa}, \quad \mu_i^* = \frac{\mu_i}{\kappa}, \\
R_i^2 &= \frac{R_i^2}{L_i}, \quad k_{CC}^r_i = \frac{k_{CC}^r}{L_i}, \quad k_{CC}^i_i = \frac{k_{CC}^i}{L_i}.
\end{align*}
$$

(42a) (42b)

Parameters not mentioned are unchanged. Let $x$, $x_i$, $x_i^*$ denote the state vectors of the unscaled inverter model, $i$-th inverter of the parallel system, and the reduced-order model, respectively. Permute the state vectors the same way as (18), denoting the permuted vectors as $\tilde{x}$, $\tilde{x}_i$, $\tilde{x}_i^*$. Partition the permuted state vector $\tilde{x} = [\lambda^T, \psi^T]^T$, where $\lambda = [\lambda^1, \lambda^2, \lambda^3, \lambda^4, \mu^1, \mu^2, \mu^3, \mu^4]^T$ and $\psi = [\psi^1, \psi^2, \psi^3, \psi^4]^T$. We also partition $\tilde{x}_i$ and $\tilde{x}_i^*$ the same way: $\tilde{x}_i = [\lambda_i^T, \psi_i^T]^T$, $\tilde{x}_i^* = [\lambda_i^*, \psi_i^*]^T$. Suppose the initial conditions are such that $\lambda_i(t_0) = \sum_{\ell=1}^N \psi_i(t_0) = \pi \lambda_i(t_0)$, $\psi_i(t_0) = \psi_i(t_0) = \psi(t_0)$, $\forall \ell$, and the inputs are:

$$
u_i^* = \sum_{\ell=1}^N \nu_{1, \ell} = \pi u_1, \quad \nu_i^2 = \nu_{2, \ell} = \nu_2, \forall \ell.
$$

(43)

It follows that for $t \geq t_0$:

$$
\lambda_i(t) = \sum_{\ell=1}^N \lambda_i(t) = \pi \lambda_i(t), \quad \psi_i(t) = \psi_i(t) = \psi(t), \forall \ell,
$$

(44)

if and only if the parameters of the reduced-order model are related to the unscaled inverter through (42a)-(42b).

**Proof:** Each of the inverters in the parallel system can be viewed as the aggregate of $\kappa_i$ inverters, while keeping in mind that $\kappa_i$ is not necessarily an integer. The rest of this proof is straightforward from Theorem 1.

**IV. EXPERIMENTAL VALIDATION & SIMULATION RESULTS**

In this section, we outline results from an experimental prototype and an exhaustive simulation study to demonstrate various aspects of the reduced-order model. The purpose and scope of the experiments is to demonstrate the validity and establish the accuracy of the reduced-order model (under uniform and symmetric settings) and this is done by comparing the net current injected by the parallel system of inverters in hardware to the output current of the aggregated reduced-order model. The experiments also establish robustness of the reduced-order model to parametric variations that are indeed inescapable in any hardware setup. Following the experimental results, we also include an exhaustive simulation study that: validates the reduced-order model derived for heterogeneous settings (Corollaries 1 and 2), investigates robustness of the reduced-order model to variations in filter parameters, and demonstrates the computational benefits of the reduced-order model.
irradiance transients that a microinverter system might contend with. The reactive-power steps are representative of, e.g., ancillary services that grid-connected inverters may provide. Results are plotted in Fig. 5. The plot pair in each subfigure illustrates:

1) The measured net sinusoidal current injected into the grid by the parallel inverters overlaid with the simulated current from the aggregated-inverter model. The measurement point and corresponding point in the reduced-order model are marked prominently in Fig. 4.

2) Pertinent $d$- or $q$-axis current waveforms measured at each inverter output in addition to measured and simulated net current injection.

It is worth emphasizing that we focus just on the net current at the point of grid interconnection and compare that with the current suggested by the aggregate model. The match between these through a variety of large-signal changes—as suggested in Fig. 5—validates the accuracy of the aggregate model. Furthermore, note that in this case, the parallel collection of inverters are collectively described by a 48-state model, while the simulations are performed with the reduced-order 16-state model.

C. Simulation Study

Next, we establish the accuracy and computational benefits of the proposed reduced-order model (for a system of 100 parallel-connected inverters) in heterogeneous settings with numerical simulation results. The parameters of the inverter with nominal power ratings are listed in Table I. We consider the following cases: #1) Inverters have heterogeneous power ratings with power-scaling parameters $\kappa$ selected to be uniformly distributed between 0.5 and 5. #2) All inverters have ratings that match the nominal power ratings, but their LCL-filter parameters vary between $\pm 10\%$ of their nominal values. #3) Same setup as #2, but the LCL-filter parameters of the inverters vary between $\pm 80\%$ of their nominal values. For all cases, the real- and reactive-power setpoints of the inverters are assumed to be uniformly distributed between $0 - 200$ W and $0 - 100$ VAR, respectively, and we perform a step change to both setpoints, with the values again selected to be uniformly distributed between $400 - 600$ W and $300 - 500$ VAR, respectively. The step change is introduced at $t = 2$ s, and we stop the simulations at $t = 4$ s. We note that case #2 and #3 have the same reduced-order model. The parameter scalings of the reduced-order models for case #1 and #2 (#3) are given by (42a)–(42b) and (17a)–(17b), respectively. The net current injection of the multi-inverter system and the reduced-order models for cases #1, #2, and #3 are shown in Fig. 6. We can clearly see in Fig. 6a that for case #1, the output current of the reduced-order model is exactly the same as the net current injection of the parallel system—this validates Corollaries 1 and 2. Furthermore, Fig. 6b shows that the reduced-order model is quite robust with respect to the parametric variations in the LCL filter parameters with discrepancies obvious in high-frequency content. For larger variation ($\pm 80\%$), Fig. 6c shows that the reduced-order model captures the dynamics of the multi-inverter system, albeit with degraded accuracy during

<table>
<thead>
<tr>
<th>Tables</th>
<th>Inverter LCL-Filter and Controller Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{\text{c}}$</td>
<td>$1.0 \text{ mH}$</td>
</tr>
<tr>
<td>$R_{\text{c}}$</td>
<td>$0.2 \text{ mH}$</td>
</tr>
<tr>
<td>$C_{\text{c}}$</td>
<td>$24 \text{ \mu F}$</td>
</tr>
<tr>
<td>$k_{\text{Pc}}$</td>
<td>$6 \text{ V/A}$</td>
</tr>
<tr>
<td>$k_{\text{Pc}}$</td>
<td>$0.01 \text{ A/V/VA}$</td>
</tr>
<tr>
<td>$k_{\text{Pl}}$</td>
<td>$0.1 \text{ A/(V/VA)}$</td>
</tr>
<tr>
<td>$\omega_{\text{c,PLL}}$</td>
<td>$50.26 \text{ rad/s}$</td>
</tr>
<tr>
<td>$\omega_{\text{P,PLL}}$</td>
<td>$10 \text{ rad/(V \cdot s)}$</td>
</tr>
<tr>
<td>$\omega_{\text{PLL}}$</td>
<td>$1.25 \text{ rad/(V \cdot s)}$</td>
</tr>
</tbody>
</table>

Fig. 5. Comparison of experimentally measured and simulated waveforms: (a) Real-power step up $p^{*}: 30$ W $\rightarrow$ 600 W with fixed $q^{*} = 0$ VAR, (b) Real-power step down $p^{*}: 700$ W $\rightarrow$ 50 W with fixed $q^{*} = 0$ VAR, (c) Reactive-power step up $q^{*}: 0$ VAR $\rightarrow$ 500 W with fixed $p^{*} = 200$ W, (d) Reactive-power step down $q^{*}: 500$ VAR $\rightarrow$ 0 W with fixed $p^{*} = 250$ W.
inverters during large-signal transients, and simulation results were provided to demonstrate computational benefits and robustness to parametric variations. Directions for future work include analytically establishing error bounds on the trajectories returned by reduced-order models in the face of parametric variations in the filter and control parameters.

APPENDIX

A. Steady-State Operation of PLL

Express the grid voltage as \( v_g = V_g \sin(\delta_g) \), where \( V_g \) and \( \delta_g \) are the voltage amplitude and angle, respectively. The corresponding \( \alpha/\beta \) components of \( v_g \) are given by

\[
\begin{align*}
  v^\alpha_g &= v_g = V_g \sin(\delta_g), \\
  v^\beta_g &= V_g \sin(\delta_g - \frac{\pi}{2}) = -V_g \cos(\delta_g).
\end{align*}
\]

The \( \delta \)-axis component of \( v_g \) is obtained from (2) as

\[
v^\delta_g = V_g \cos(\delta) \sin(\delta_g) - V_g \sin(\delta) \cos(\delta_g) = V_g \sin(\delta_g - \delta).
\]

From above, it follows that when \( \delta = \delta_g \), \( v^\delta_g = 0 \).

B. State-Space Model Particulars

The state-space model of the PLL is

\[
\begin{bmatrix}
  -\frac{R_i}{L_i} & 0 & 0 & 0 \\
  0 & 0 & \omega_{nom} & 0 \\
  0 & \omega_{nom} & 0 & 0 \\
  -\frac{R_i}{L_i} & 0 & 0 & \omega_{nom}
\end{bmatrix}
\]

\[
A_{LCL} = \begin{bmatrix}
  -\frac{R_i}{L_i} & 0 & 0 & 0 \\
  0 & 0 & \omega_{nom} & 0 \\
  0 & \omega_{nom} & 0 & 0 \\
  -\frac{R_i}{L_i} & 0 & 0 & \omega_{nom}
\end{bmatrix},
\]

\[
A_{CC} = \begin{bmatrix}
  0 & -k_{PC}^p & 0 & 0 \\
  -k_{PC}^p & 0 & 0 & 0 \\
  0 & 0 & 0 & k_{PC}^p \\
  0 & 0 & -k_{PC}^p & 0
\end{bmatrix},
\]

\[
A_{PC} = \begin{bmatrix}
  -\omega_{PC} & 0 & 0 & 0 \\
  0 & -\omega_{PC} & 0 & 0 \\
  -1 & 0 & 0 & 0 \\
  0 & -1 & 0 & 0
\end{bmatrix},
\]

\[
B_{CC} = \begin{bmatrix}
  0 & k_{PC}^p & 0 & 0 \\
  -k_{PC}^p & 0 & 0 & 0 \\
  0 & 0 & 0 & k_{PC}^p \\
  0 & 0 & -k_{PC}^p & 0
\end{bmatrix},
\]

\[
A_{PLL} = \begin{bmatrix}
  -\omega_{PLL} & 0 & 0 & 0 \\
  0 & -\omega_{PLL} & 0 & 0 \\
  -1 & 0 & 0 & 0 \\
  0 & -1 & 0 & 0
\end{bmatrix},
\]

the transient. Finally, the computation time for the 1600-th order multi-inverter system simulation for cases #1 and #2 are 58.23 s, 66.97 s, and 145.08 s, respectively, and of the reduced-order 16-th order aggregate model are 1.89 s, 1.62 s, and 1.62 s, respectively. This clearly establishes the computational benefits of the proposed model.

V. CONCLUDING REMARKS AND DIRECTIONS FOR FUTURE WORK

In this paper, we derived a reduced-order aggregated model for identical parallel-connected grid-tied single-phase inverters and extensions covering cases when the inverter power-setpoints are different and the inverter power ratings are different. The reduced-order model preserves the structure and has the same order as any individual inverter in the parallel collection. Experimental validation was provided to establish the accuracy of the reduced-order model in capturing ac-side dynamics of
Lastly, the entries of \( g(x, u_1, u_2) \), with \( g_\ell \) denotes the \( \ell \)-th entry of \( g(x, u_1, u_2) \), are

\[
g_1 = \frac{1}{L_i} (k_i^\text{PC}(i_1^\alpha - i_1^\beta \cos \delta - i_1^\beta \sin \delta) + k_i^\text{CC} \gamma_i^d) \cos \delta
\]

\[
- \frac{1}{L_i} (k_i^\text{PC}(i_1^\alpha - i_1^\beta \cos \delta - i_1^\beta \sin \delta) + k_i^\text{CC} \gamma_i^d) \sin \delta,
\]

\[
g_2 = \eta(i_1^\alpha - i_1^\beta) - g_1, \quad g_3 = 0, \quad g_4 = 0,
\]

\[
g_5 = R_i g_1, \quad g_6 = \eta(v_1^\alpha - v_1^\beta) - g_5,
\]

\[
g_7 = -i_1^\alpha \cos \delta - i_1^\beta \sin \delta, \quad g_8 = i_1^\alpha \sin \delta - i_1^\beta \cos \delta,
\]

\[
g_9 = \frac{\omega_e \cdot p_{\text{PC}}}{2} (v_1^\alpha, v_1^\beta, v_2^\alpha, v_2^\beta), \quad g_{10} = \frac{\omega_e \cdot p_{\text{PC}}}{2} (v_1^\alpha, v_1^\beta - v_2^\alpha, v_2^\beta),
\]

\[
g_{11} = 0, \quad g_{12} = 0, \quad g_{13} = \eta(v_1^\alpha - v_2^\beta),
\]

\[
g_{14} = \omega_{e, p_{\text{PLL}}} (v_1^\alpha \cos \delta + v_2^\beta \sin \delta), \quad g_{15} = 0, \quad g_{16} = \omega_{\text{nom}}
\]

where \( \eta := -k_i^\text{PLL} v_{\text{PLL}} + k_i^\text{PLL} \phi_{\text{PLL}}, \quad i_1^\alpha = k_i^\text{PC} (q_1 - q_{\text{avg}}) + k_i^\text{PC} \phi_i\), and \( i_1^\beta = k_i^\text{PC} (p_1 - p_{\text{avg}}) + k_i^\text{CC} \phi_i\).

### C. Proof of Corollary 1

We begin by noting that the PLL dynamics are decoupled, and the its parameters in the individual and reduced-order models are the same, therefore \( \forall t \geq t_0, \ x_{\text{PLL}}(t) = x_{\text{PLL}, t}(t) \) \( \forall t \) if we initialize \( x_{\text{PLL}}(t_0) = x_{\text{PLL}, t}(t_0) \) \( \forall t \). Next, partition the permuted versions of (12) and (16), excluding the PLL dynamics, as

\[
\begin{bmatrix}
\dot{\lambda}_t \\
\dot{i}_t^\alpha, i_t^\beta, i_t^\alpha, i_t^\beta, i_{\ell, t}^\alpha, i_{\ell, t}^\beta, i_{\ell, t}^\alpha, i_{\ell, t}^\beta
\end{bmatrix} =
\begin{bmatrix}
\hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13} & \hat{A}_{14} & \hat{B}_{11} & \hat{B}_{12} & \hat{B}_{13} & \hat{B}_{14} \\
\hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} & \hat{A}_{24} & \hat{B}_{21} & \hat{B}_{22} & \hat{B}_{23} & \hat{B}_{24} \\
\hat{A}_{31} & \hat{A}_{32} & \hat{A}_{33} & \hat{A}_{34} & \hat{B}_{31} & \hat{B}_{32} & \hat{B}_{33} & \hat{B}_{34} \\
\hat{A}_{41} & \hat{A}_{42} & \hat{A}_{43} & \hat{A}_{44} & \hat{B}_{41} & \hat{B}_{42} & \hat{B}_{43} & \hat{B}_{44}
\end{bmatrix}
\begin{bmatrix}
\lambda_t \\
i_t^\alpha, i_t^\beta, i_{\ell, t}^\alpha, i_{\ell, t}^\beta, i_{\ell, t}^\alpha, i_{\ell, t}^\beta
\end{bmatrix}
+ \begin{bmatrix}
\hat{g}_{11}(\hat{x}, \hat{u}_1, \hat{u}_2) \\
\hat{g}_{12}(\hat{x}, \hat{u}_1, \hat{u}_2) \\
\hat{g}_{13}(\hat{x}, \hat{u}_1, \hat{u}_2) \\
\hat{g}_{14}(\hat{x}, \hat{u}_1, \hat{u}_2)
\end{bmatrix}
\]

\[
\begin{bmatrix}
\dot{\lambda}_{\ell, t} \\
\dot{i}_{\ell, t}^\alpha, i_{\ell, t}^\beta
\end{bmatrix} =
\begin{bmatrix}
\hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13} & \hat{A}_{14}
\hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} & \hat{A}_{24}
\hat{A}_{31} & \hat{A}_{32} & \hat{A}_{33} & \hat{A}_{34}
\hat{A}_{41} & \hat{A}_{42} & \hat{A}_{43} & \hat{A}_{44}
\end{bmatrix}
\begin{bmatrix}
\lambda_{\ell, t} \\
i_{\ell, t}^\alpha, i_{\ell, t}^\beta
\end{bmatrix}
+ \begin{bmatrix}
\hat{g}_{11}(\hat{x}, \hat{u}_1, \hat{u}_2) \\
\hat{g}_{21}(\hat{x}, \hat{u}_1, \hat{u}_2)
\end{bmatrix}
\]

where \( \hat{g}_{11}, \hat{g}_{12}, \hat{g}_{13}, \hat{g}_{14} : \mathbb{R}^{16} \times \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^{10} \) and \( \hat{g}_{21} : \mathbb{R}^{16} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) are the nonlinear parts of the dynamics of \( \lambda_t \) and \( i_{\ell, t}^\alpha, i_{\ell, t}^\beta, i_{\ell, t}^\alpha, i_{\ell, t}^\beta \), respectively (similarly for \( \hat{g}_{21} \) and \( \hat{g}_{22} \)). We bring to note a slight abuse of notation in terms of the submatrices in (45) and (46) and those in (22) and (23). Furthermore, the submatrices in (45) and (46) also follow the relationships in (24). Define \( z_1 := \lambda_t - \sum_{\ell=1}^N \lambda_{\ell, t} \) and \( z_2 := \sum_{\ell=1}^N i_{\ell, t}^\alpha, i_{\ell, t}^\beta, i_{\ell, t}^\alpha, i_{\ell, t}^\beta \). The dynamics of \( z_1 \) and \( z_2 \) are:

\[
\dot{z}_1 = \dot{\lambda}_t - \sum_{\ell=1}^N \lambda_{\ell, t} = A_{11} \lambda_t + A_{12} i_{\ell, t}^\alpha, i_{\ell, t}^\beta, i_{\ell, t}^\alpha, i_{\ell, t}^\beta + B_{11} u_1 + B_{12} u_2,
\]

\[
\dot{z}_2 = \dot{\lambda}_{\ell, t} - \sum_{\ell=1}^N i_{\ell, t}^\alpha, i_{\ell, t}^\beta, i_{\ell, t}^\alpha, i_{\ell, t}^\beta = A_{11} \lambda_t + A_{12} i_{\ell, t}^\alpha, i_{\ell, t}^\beta, i_{\ell, t}^\alpha, i_{\ell, t}^\beta + B_{11} u_1 + B_{12} u_2,
\]

Next, we will show that

\[
\hat{g}_{11}(\hat{x}, \hat{u}_1, \hat{u}_2) - \sum_{\ell=1}^N \hat{g}_{11}(\hat{x}, \hat{u}_1, \hat{u}_2) = \hat{g}_{11}(\chi, 0, 2u_2),
\]

\[
N \hat{g}_{21}(\hat{x}, \hat{u}_1, \hat{u}_2) - \sum_{\ell=1}^N \hat{g}_{21}(\hat{x}, \hat{u}_1, \hat{u}_2) = \hat{g}_{21}(\chi, 0, 2u_2),
\]
\[ \begin{align*}
&\left( k_{PC}^p \right) \left( k_{PC}^p \right) \left( 0 - \left( q_{avg} - \sum_{l=1}^{N} q_{avg,l} \right) \right) + k_{PC}^{g} \left( R_{PC}^{\varphi} \right) \\
&- \sum_{l=1}^{N} \phi_l \left( q_{i,l}^{\alpha} \right)^2 \\
&+ \left( k_{PC}^p \right) \left( k_{PC}^p \right) \left( 0 - \left( p_{avg} - \sum_{l=1}^{N} p_{avg,l} \right) \right) + k_{PC}^{g} \left( \phi_i^{\alpha} \right) \\
&- \sum_{l=1}^{N} \phi_l \left( \sum_{l=1}^{N} q_{l}^{\alpha} \right) - \left( \sum_{l=1}^{N} q_{l}^{\alpha} \right) \sin \delta \\
&- \left( \sum_{l=1}^{N} q_{l}^{\alpha} \right) \cos \delta \\
&= \tilde{g}_{1,1}(\chi, 0_2, u_2),
\end{align*} \]

\[ \begin{align*}
&\tilde{g}_{1,2}(\tilde{x}) - \sum_{l=1}^{N} \tilde{g}_{1,2}(\tilde{x}) = \left( -k_{PLL}^{\varphi} v_{PLL}^p + k_{PLL}^{\varphi} \phi_{PLL}^p \right)(i_{\alpha}^{\alpha}) \\
&- i_{\beta}^{\alpha} - \tilde{g}_{1,1}(\tilde{x}) - \sum_{l=1}^{N} \left( -k_{PLL}^{\varphi} v_{PLL}^{\beta} + k_{PLL}^{\varphi} \phi_{PLL}^{\beta} \right)(i_{\alpha}^{\beta}) \\
&- i_{\beta}^{\beta} - \tilde{g}_{1,1}(\tilde{x}) = \left( -k_{PLL}^{\varphi} v_{PLL}^{\beta} + k_{PLL}^{\varphi} \phi_{PLL}^{\beta} \right)(i_{\alpha}^{\beta}) \\
&- \left( \sum_{l=1}^{N} (i_{\alpha}^{\beta} - \sum_{l=1}^{N} i_{\alpha}^{\beta} \right) - \left( \tilde{g}_{1,1}(\tilde{x}) \right)
\end{align*} \]

\[ \begin{align*}
&\tilde{g}_{1,3}(\tilde{x}) - \sum_{l=1}^{N} \tilde{g}_{1,3}(\tilde{x}) = 0 = \tilde{g}_{1,3}(\chi, 0_2, u_2),
\end{align*} \]

\[ \begin{align*}
&\tilde{g}_{1,4}(\tilde{x}) - \sum_{l=1}^{N} \tilde{g}_{1,4}(\tilde{x}) = 0 = \tilde{g}_{1,4}(\chi, 0_2, u_2),
\end{align*} \]

\[ \begin{align*}
&\tilde{g}_{1,5}(\tilde{x}) - \sum_{l=1}^{N} \tilde{g}_{1,5}(\tilde{x}) = -i_{\alpha}^{\alpha} \cos \delta - i_{\beta}^{\alpha} \sin \delta \\
&- \left( \sum_{l=1}^{N} (i_{\alpha}^{\beta} \cos \delta - i_{\beta}^{\alpha} \sin \delta) \right) - \left( \sum_{l=1}^{N} i_{\alpha}^{\beta} \cos \delta \right) \\
&- \left( i_{\alpha}^{\alpha} - \sum_{l=1}^{N} i_{\alpha}^{\beta} \sin \delta \right) \sin \delta = \tilde{g}_{1,5}(\chi, 0_2, u_2),
\end{align*} \]

\[ \begin{align*}
&\tilde{g}_{1,6}(\tilde{x}) - \sum_{l=1}^{N} \tilde{g}_{1,6}(\tilde{x}) = i_{\alpha}^{\alpha} \sin \delta - i_{\beta}^{\alpha} \cos \delta \\
&- \left( \sum_{l=1}^{N} i_{\alpha}^{\beta} \sin \delta - i_{\beta}^{\alpha} \cos \delta \right) = \left( i_{\alpha}^{\alpha} - \sum_{l=1}^{N} i_{\alpha}^{\beta} \sin \delta \right) \sin \delta \\
&- \left( i_{\alpha}^{\alpha} - \sum_{l=1}^{N} i_{\alpha}^{\beta} \sin \delta \right) \cos \delta = \tilde{g}_{1,6}(\chi, 0_2, u_2),
\end{align*} \]
\[ g_2(\chi, 0_2, u_2) = 0 \] when \( z_1 = 0_1 \) and \( z_2 = 0_2 \). By the definition of \( z_1 \) and \( z_2 \), we have
\[ \chi^2(t) = \sum_{t} \chi(t), u_i^{\alpha_\beta_\gamma}(t) = \frac{1}{N} \sum_{i=1}^{N} u_i^{\alpha_\beta_\gamma}(t), \forall t \geq t_0. \]

For the other direction, given that \( \forall t \geq t_0; z_1(t) = \chi^2(t) - \sum_{i=1}^{N} u_i^{\alpha_\beta_\gamma}(t) = 0_1, x_{PLL}(t) = x_{PLL}(t), \forall t \in (47) \) and (48) can be written as
\[ 0_1 = (\hat{A}_{11} - \hat{A}_{12})\chi_1 + (\hat{A}_{12} - N\hat{A}_{12})u_1^{\alpha_\beta_\gamma} + (\hat{B}_{11} - \hat{B}_{12})u_1 + (\hat{B}_{12} - N\hat{B}_{12})u_2 + \hat{g}_1(\hat{x}_1, u_1, u_2) - \sum_{i=1}^{N} \hat{g}_1(\hat{x}_i, u_1, u_2). \]

These equalities are satisfied when the following identities hold:
\[ \hat{A}_{11} = \hat{A}_{11}, \hat{A}_{12} = N\hat{A}_{12}, \hat{A}_{21} = \frac{1}{N}\hat{A}_{21}, \hat{A}_{22} = \hat{A}_{22}, \]
\[ \hat{B}_{11} = \hat{B}_{11}, \hat{B}_{21} = N\hat{B}_{21}, \hat{B}_{12} = \frac{1}{N}\hat{B}_{12}, \hat{B}_{22} = \hat{B}_{22}, \]
\[ g_1(\hat{x}_1, u_1, u_2) = \sum_{i=1}^{N} \hat{g}_1(\hat{x}_i, u_1, u_2), \]
\[ N\hat{g}_2(\hat{x}_1, u_1, u_2) = \sum_{i=1}^{N} \hat{g}_2(\hat{x}_i, u_1, u_2). \]

It is straightforward to see that (17a) and the unscaled parameters are the only set of parameters that satisfy (65). For the rest of the parameters, i.e., \( \hat{R}_1, L_1, b_1^p \), and \( k_1^{CC} \), it can be derived that they always appear in (65)–(67) as fractions of \( \frac{1}{N} \), \( \frac{1}{L_1} \), and \( \frac{1}{k_1^{CC}} \). Therefore, they are related to those in the reduced-order model through (17b). This concludes the proof.

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