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Phase Balancing in Globally Connected Networks of Liénard Oscillators

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***Abstract*— We synthesize a feedback for a fully connected network of identical Liénard-type oscillators such that the phase-balanced equilibrium—the mode where the centroid of the coupled oscillators in polar coordinates is at the origin—is asymptotically stable, and the phase-synchronized equilibrium is unstable. Our approach hinges on a coordinate transformation of the oscillator dynamics to polar coordinates, and periodic averaging theory to simplify the examination of multiple time-scale behavior. Using Lyapunov- and linearization-based arguments, we demonstrate that the oscillator dynamics have the same radii and balanced phases in steady state for a large set of initial conditions. Numerical simulation results are presented to validate the analyses.**

I. Introduction

Collective motion of oscillators has been a widely studied problem in various disciplines spanning neuroscience, engineering, and physics [1]–[8]. With emphasis to engineering applications, designing feedback control laws such that multiple agents achieve a desired formation is relevant in problems such as coordination of unmanned autonomous vehicles [9], control of vehicle platoons [10], [11], and synchronization of inverters in electrical networks [12], [13].

In this paper, we study three types of equilibria for a fully-connected system of Liénard-type oscillators in the quasi-harmonic regime: the phase-balanced state, the phase synchronous state, and the bi-cluster synchronous state. To simplify exposition of these different equilibria, consider: i) a collection of N oscillators (indexed in the set $\mathcal{N}$) with phases $\theta_1, \ldots, \theta_N$; and ii) the *order parameter*, $R\mathrm{e}^{\jmath\psi} = \frac{1}{N}\sum_{k=1}^{N} \mathrm{e}^{\jmath\theta_k}$, a metric that quantitatively captures phase cohesiveness [7], [14], [15] and represents the centroid of all oscillators (when conceptualized to be points on the circle). The magnitude of the order parameter is a synchronization measure [7]. The case where $R = 0$ corresponds to the phase-balanced state, i.e., the phases $\theta_1, \ldots, \theta_N$ are spaced apart such that $\sum_{k=1}^{N} \mathrm{e}^{\jmath\theta_k} = 0$ [7]. The case where $R = 1$ corresponds to the phase-synchronized state, i.e., the phases $\theta_1, \ldots, \theta_N$ are such that $\theta_\ell = \theta_k, \forall \ell, k \in \mathcal{N}$. One phase-balanced set of particular interest is the splay state where the phases are uniformly distributed around the circle, i.e., $\theta_k = k\frac{2\pi}{N} + \phi \pmod{2\pi}, k \in \mathcal{N}, 0 \leq \phi \leq 2\pi$ [16], [17]. Finally, the bi-cluster synchronous state refers to motion with phases evolving in one of two phase-synchronized clusters, i.e., $\theta_\ell = \theta_k \pmod{\pi}, \forall \ell, k \in \mathcal{N}$. Perfect phase synchronization of Liénard-type oscillators (and dynamical systems in general) has been widely studied (see [7], [18] for detailed surveys). In particular, the diffusive interconnection, which guarantees synchronization, corresponds to a feedback which uses the graph Laplacian [19] as the feedback gain matrix. Altering the signs of the feedback—which we attempt in this work—ensures the stability of the phase-balanced state instead [9], [17], [20]. Splay states have also been investigated for various oscillator models like Kuramoto oscillators [21], [22], kinematic models [9], [23], and Van der Pol oscillators [17], [20] (which fall in the class of Liénard oscillators considered here). We devote attention to the phase balanced state which contain the splay states, since this is pertinent to several engineering systems such as autonomous underwater vehicles; where coordinated, periodic trajectories can be used to collect data with requisite spatial and temporal separation as remarked in [9], or in the control for collective circular motion of nonholonomic vehicles [24]. Another anticipated application is in networks of dc-dc converters, where carrier wave interleaving, which refers to the temporal separation of the triangular waves used for PWM, minimizes current ripple and harmonics. Using Liénard-type oscillators to locally construct triangular waves of the same phase for each converter, we can guarantee that the carrier waves for the converters are interleaved just by virtue of the electrical network interaction.

A short description of our approach is provided next. We begin by defining a state-space model for the coupled oscillators which we transform from Euclidean to polar coordinates. Inspired by [25] and following our earlier work [26], we make use of periodic averaging to obtain an autonomous system. We then construct a Lyapunov function to ascertain the convergence of the oscillator dynamics to the phase-balanced state. Additionally, we show that all those solutions that do not belong to the phase-balanced state are locally unstable. In the spirit of [23], our work delineates a rigorous Lyapunov analysis for an all-to-all coupling network and, in doing so, provides a system theoretic understanding to the observations in [3], [17], [20] which leveraged symmetry and equivariant bifurcation theory to study the dynamics of identical dissipative oscillators. Sufficient conditions for anti-phase synchronization for a two-oscillator case for Van der Pol oscillators has also been reported in [27] by using contraction analysis; our work subsumes the result and provides an analysis of globally coupled N Liénard-type oscillators. Furthermore, our approach also improves upon the

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methods used in [17] where linearization- and bifurcation-based arguments are leveraged to explain the splay states observed in coupled Van der Pol oscillators by providing almost global guarantees for convergence to the solutions and generalizing it to a class of Liénard-type oscillators.

The remainder of this paper is organized as follows. Section II introduces the mathematical preliminaries and the coupled nonlinear oscillator model. Building upon this, Section III outlines the nature of the solutions of the system and the stability of possible equilibria. We validate our analysis through numerical simulations in Section IV and conclude with some suggestions for future work in Section V.

II. Preliminaries

We first outline notation used in the manuscript. We then discuss the oscillator model and provide an overview of periodic averaging that is leveraged in the analysis.

A. Notation

By way of notation, $\jmath := \sqrt{-1}$, z^* denotes the complex conjugate of $z \in \mathbb{C}$ and $\|\cdot\|_2$ denotes the Euclidean norm of a complex vector. The N-dimensional space of nonnegative reals is denoted by $\mathbb{R}^N_{\geq 0}$, $\mathbb{Z}$ represents the set of integers, $\mathbb{C}^N$ denotes the N-dimensional space of complex numbers, and $\mathbb{T}^N$, the N-dimensional torus. Also, for a matrix X, $\left[X\right]_{jk}$ represents the entry in its jth row and kth column, $\mathrm{Ker}(X)$ denotes the kernel of the matrix and $\mathrm{Ker}_{\mathbb{R}}(X)$ denotes intersection of $\mathrm{Ker}(X)$ with $\mathbb{R}^N$. For vector x, $\mathrm{diag}\{x\}$ denotes the diagonal matrix obtained by stacking elements of x on the main diagonal. Finally, 1_N and 0_N denote vectors of all ones and all zeros; e_j is the unit basis vector with 1 at the jth place and zeros elsewhere; and I_N, the $N \times N$ identity matrix.

B. Nonlinear Oscillator Model and Coupling Network

Liénard-type oscillators are described by the nonlinear second-order differential equation of the general form [28]

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ (respectively, $g : \mathbb{R} \to \mathbb{R}$) is an even (respectively, odd) and continuously differentiable function. We will frequently reference the function

$$h(x) := \int_{\tau=0}^{x} f(\tau) d\tau, \tag{2}$$

which accordingly is odd and satisfies the Liénard theorem criterion, i.e., it has exactly one positive zero at $x = \eta > 0$, is strictly negative for $0 < x < \eta$, is positive and nondecreasing for $x > \eta$, and $h(x) \to \infty$ as $x \to \infty$.

For the coupled system, we consider the following forced Liénard-type oscillator dynamics

$$\ddot{x} + \varepsilon f(x)\dot{x} + \omega^2 x = \varepsilon \dot{u}, \tag{3}$$

where u is the input, and ω and ε are positive constants. Since we are interested in near-sinusoidal oscillations, we confine ε to the limits $0 < \varepsilon \ll 1$. As $\varepsilon \to 0$, we see that the unforced version of (3) reduces to a simple harmonic oscillator with resonant frequency ω. In addition to this standard setting for Liénard-type oscillators, we make the following assumption on the function $h(x)$.

Assumption 1. *$h(x) = \frac{N}{2}x$ admits a unique positive solution, ρ, and $\left(\int_0^x h\left(s\right) ds - cx^2\right) \to \infty$ as $x \to \infty$ for any real scalar $c \in \mathbb{R}$.*

This is a mild restriction and is satisfied, e.g., for the Van der Pol oscillator for which $h(x) = x^3/3 - x$.

We study the collective motion of $N \geq 2$ identical Liénard-type oscillators described by dynamics (3) indexed by elements in the set $\mathcal{N} := \{1, \ldots, N\}$. We will find it useful to transcribe the dynamics of the jth oscillator (3) using $y_j(t) = x_j(t)$ and $z_j(t) = \omega \int_0^t x_j(\tau) d\tau$ as states:

$$\dot{z}_j = \omega y_j, \quad \dot{y}_j = -\omega z_j - \varepsilon h(y_j) + \varepsilon u_j\,, \tag{4}$$

where $h_j : \mathbb{R} \to \mathbb{R}$ is defined in (2). The oscillators are connected over an undirected graph $\mathcal{G}$, and we assume that the graph is complete and without self loops. Denoting $u = [u_1, \ldots, u_N]^{\mathrm{T}}$ and $y = [y_1, \ldots, y_N]^{\mathrm{T}}$, the interactions between the oscillators are captured by the *positive* (and thus repulsive) diffusive coupling

$$u = Ly, \tag{5}$$

where $L = NI_N - 1_N 1_N^{\mathrm{T}}$ is the Laplacian matrix of $\mathcal{G}$.

C. Averaging Theory

In the parametric regime $0 < \varepsilon \ll 1$, the dynamical behavior of individual oscillators is weakly nonlinear, and they are weakly coupled. This results in multiple time-scale behavior, the analysis of which can be significantly simplified with periodic averaging [29]. We describe this briefly next. Consider a time-varying dynamical system

$$\dot{x} = \varepsilon p(x, t, \varepsilon), \tag{6}$$

with time-periodic vector field $p(x, t, \varepsilon) = p(x, t+T, \varepsilon)$ with period $T > 0$, and $0 < \varepsilon \ll 1$. The associated *time-averaged dynamical system* is given by

$$\dot{\overline{x}} = \varepsilon \overline{p}(\overline{x}) = \varepsilon \frac{1}{T} \int_{\tau=0}^{T} p(\overline{x}, \tau, 0) d\tau. \tag{7}$$

The solution of the averaged system (7) in the time scale εt is $\mathcal{O}(\varepsilon)$ *close* to the solution of the original system (6), i.e.,

$$||x(t, \varepsilon) - \overline{x}(\varepsilon t)||_2 = \mathcal{O}(\varepsilon), \quad \forall t \in [0, t^*], \tag{8}$$

for some $t^* > 0$ for which unique solutions exist for both (6) and (7), and assuming $||x(0, \varepsilon) - \overline{x}(0)||_2 = \mathcal{O}(\varepsilon)$. This lets us work with *averaged* quantities without compromising on accuracy while inferring the dynamics of the original system.

III. Nature and Stability of Collective Motion

In this section, we characterize the nature of the trajectories of the coupled Liénard oscillators and study the stability of a few equilibria of interest. We begin by leveraging a coordinate transformation to polar coordinates and the weakly nonlinear property of the system to develop an averaged model for our system to facilitate analysis.

A. Averaged Model

Consider the following bijective coordinate transformation from the state-space model in (4) to polar coordinates:

$$z_j \ \to r_j \sin(\omega t + \theta_j), \quad y_j \ \to r_j \cos(\omega t + \theta_j). \tag{9}$$

In these new set of coordinates, the amplitude dynamics of the j oscillator are given by:

$$\begin{aligned}\dot{r}_j &= -\varepsilon h(r_j \cos(\omega t + \theta_j)) \cos(\omega t + \theta_j) \\ &\quad - \varepsilon \sum_{k=1}^{N} (r_k \cos(\omega t + \theta_k)) \cos(\omega t + \theta_j),\end{aligned} \tag{10}$$

and the phase dynamics are given by

$$\begin{aligned}\dot{\theta}_j &= \frac{\varepsilon}{r_j} h(r_j \cos(\omega t + \theta_j)) \sin(\omega t + \theta_j) \\ &\quad + \frac{\varepsilon}{r_j} \sum_{k=1}^{N} (r_k(\cos \omega t + \theta_k)) \sin(\omega t + \theta_j).\end{aligned} \tag{11}$$

Using the averaging operation in (7), we can approximate the amplitude and phase dynamics of the non-autonomous system above in (10)–(11) with the following autonomous system (we omit the simple but lengthy integral calculations)

$$\begin{aligned}\dot{\overline{r}}_j &= -\varepsilon \overline{h}(\overline{r}_j) + \frac{\varepsilon(N-1)}{2} \overline{r}_j - \frac{\varepsilon}{2} \sum_{k=1, k\neq j}^{N} \overline{r}_k \cos(\overline{\theta}_{jk}), \\ \dot{\overline{\theta}}_j &= \frac{\varepsilon}{2\overline{r}_j} \sum_{k=1, k\neq j}^{N} \overline{r}_k \sin(\overline{\theta}_{jk}),\end{aligned} \tag{12}$$

where we define $\overline{\theta}_{jk} := \overline{\theta}_j - \overline{\theta}_k$ for notational convenience. The dynamics in (12) allow us to restrict attention to the domain $\overline{r}_j > 0$, $\forall j \in \mathcal{N}$. This is formalized next.

Proposition 1. *The set*

$$\mathcal{I} := \{(\overline{r}, \overline{\theta}) \in \mathbb{R}^N_{\geq 0} \times \mathbb{T}^N : \overline{r}_j > 0, \forall j \in \mathcal{N}\} \tag{13}$$

is positively invariant under the flow (12)*.*

Proof. The state-space model (4) in Euclidean coordinates can be extended to the following interconnected system:

$$\dot{z} = \omega y\,;\ \dot{y} = -\omega z - \varepsilon \Gamma(y) + \varepsilon L y\,, \tag{14}$$

where $z = [z_1, \ldots, z_N]^{\mathrm{T}}$ and $y = [y_1, \ldots, y_N]^{\mathrm{T}}$ collect the states of the N connected oscillators, and $\Gamma(y) := [h(y_1), \ldots, h(y_N)]^{\mathrm{T}}$, with $h(\cdot)$ defined in (2). The Jacobian of (14) around the origin is denoted by J_0 and it is given by:

$$J_0 := \left[\begin{array}{c|c} 0 & \omega I_N \\ \hline -\omega I_N & \Lambda \end{array}\right], \tag{15}$$

where $\Lambda := -\varepsilon \Gamma'(0) + \varepsilon L$ is a diagonally dominant matrix with positive diagonal entries and is therefore positive definite. Let the set $\{\lambda_1, \ldots, \lambda_N\}$ denote eigenvalues of Λ. Then, $2N$ eigenvalues of J_0 can be written as $0.5\left(\lambda_j \pm \sqrt{\left(\lambda_j^2 - 4\omega^2\right)}\right)$, $j \in \mathcal{N}$. Thus, the real parts of the eigenvalues of the Jacobian at the origin are positive and therefore the origin does not have a stable manifold and is repulsive [30]. Now, going back to the averaged polar coordinates, recall that

$$\overline{r}_j(t) = \frac{\omega}{2\pi} \int_{t-2\pi/\omega}^{t} r_j(\tau) d\tau\,. \tag{16}$$

Since $r_j(t)$ is a nonnegative and $2\pi/\omega$-periodic quantity, $\overline{r}_j = 0$ if and only if there exists an interval $[t_1, t_2]$ of length greater than the period, i.e., $|t_2 - t_1| \geq 2\pi/\omega$ such that $r_j(t) = 0$ for $t \in [t_1, t_2]$. As the trajectories with $r_j(t) = 0$ are unstable, we exclude them and infer that the set $\mathcal{I}$ is positively invariant. ■

B. Lyapunov Stability

Now, we use the averaged model in (12) to study the nature of the trajectories of the collective motion of oscillators and their stability. To this end, we begin by defining various sets in which the equilibria of the dynamics could reside.

Definition 1 (**Phase-balanced Set**)**.** *The set that describes collective motion where the centroid of the coupled oscillator system in polar coordinates is at the origin referred to as the* phase-balanced *set,* $\mathcal{S}$*:*

$$\mathcal{S} := \left\{(\overline{r}, \overline{\theta}) \in \mathcal{I} : \overline{r}_j = \overline{r}_k, \sum_{k=1}^{N} \mathrm{e}^{\jmath \overline{\theta}_k} = 0, \forall j, k \in \mathcal{N}\right\}. \tag{17}$$

This is similar to the *balanced set* defined in [4], [7]. Another set of solutions, the cluster synchronous set, is defined next.

Definition 2 (**Bi-Cluster Synchronous Set**)**.** *The set that describes collective motion in which the oscillators belong to one of two phase-synchronized clusters, which are* π *apart themselves, is called the* bi-cluster synchronous *set,* $\mathcal{S}'$*:*

$$\mathcal{S}' := \left\{(\overline{r}, \overline{\theta}) \in \mathcal{I} : \dot{\overline{r}}_j = 0 \ \ \overline{\theta}_{jk} = m\pi \ \forall j, k \in \mathcal{N}, m \in \mathbb{Z}\right\}. \tag{18}$$

Finally, we define the phase-synchronized set. Note that we are interested in stabilizing periodic orbits to the phase-balanced set, $\mathcal{S}$; and to show that with the feedback in (5), the phase synchronous state (defined next) is not stable.

Definition 3 (**Phase Synchronous Set**)**.** *The set that describes the phase-synchronized collective motion is called the* phase synchronous *set,* $\mathcal{S}''$*:*

$$\begin{aligned}\mathcal{S}'' := \big\{(\overline{r}, \overline{\theta}) \in \mathcal{I} : \overline{r}_j = \overline{r}_k,\, \overline{\theta}_{jk} = 2m\pi \\ \forall j, k \in \mathcal{N},\, m \in \mathbb{Z}\big\}.\end{aligned} \tag{19}$$

It is worth pointing out a few facts about the sets described above. First, we note that $S'' \subset S'$ and $S'' \cap S = \emptyset$. Furthermore, $S \cap S' \neq \emptyset$ if and only if the number of oscillators are even and each of the two phase-synchronized clusters in $\mathcal{S}'$ have exactly the same number of oscillators. These aspects are illustrated in Fig. 1. With the above definitions in place,

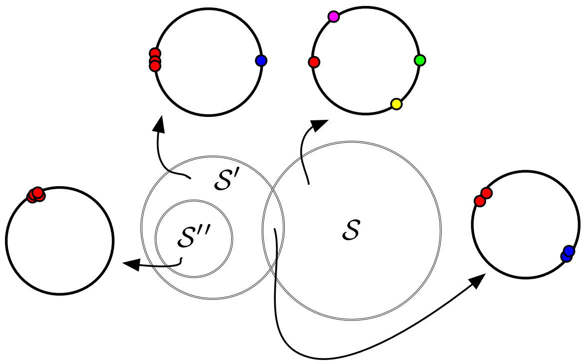


Fig. 1: Equilibria from the phase-balanced set, $\mathcal{S}$, the bi-cluster synchronous set, $\mathcal{S}'$, and the phase-synchronous set, $\mathcal{S}''$ are depicted to demonstrate the nature of the trajectories in polar coordinates. In particular, $\mathcal{S}'' \subset \mathcal{S}'$ and $\mathcal{S}'' \notin \mathcal{S}$. Furthermore, $\mathcal{S}$ and $\mathcal{S}'$ have common elements if and only if the number of oscillators are even and each of the two phase-synchronized clusters in $\mathcal{S}'$ have exactly the same number of oscillators.

we construct a Lyapunov function to establish convergence of trajectories generated by (12) starting from initial conditions in the set $\mathcal{I}$ to the phase-balanced set, $\mathcal{S}$, or the bi-cluster synchronous set, $\mathcal{S}'$.

Theorem 1. *Consider the collective motion of the N networked oscillators, with the dynamics of each described by the flow* (12). *Then, for all initial conditions* $(\overline{r}_0, \overline{\theta}_0) \in \mathcal{I}$, *trajectories converge either to* $\mathcal{S}$ (17) *or* $\mathcal{S}'$ (18).

Proof. The dynamics (12) can be written as the gradient flow:

$$\dot{\overline{r}}_j = -\nabla_{\overline{r}_j} V(\overline{r}, \overline{\theta})\,;\; \dot{\overline{\theta}}_j = -\frac{1}{\overline{r}_j^2} \nabla_{\overline{\theta}_j} V(\overline{r}, \overline{\theta}), \tag{20}$$

where $V(\overline{r}, \overline{\theta})$ is a potential function given by

$$\begin{aligned} V(\overline{r}, \overline{\theta}) = \varepsilon \Bigg(&\sum_{j=1}^{N} \int_{\tau=0}^{\overline{r}_j} \overline{h}(\tau) d\tau - \frac{N-1}{4} \sum_{j=1}^{N} \overline{r}_j^2 \\ &+ \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1, k\neq j}^{N} \overline{r}_j \overline{r}_k \cos(\overline{\theta}_{jk}) \Bigg). \end{aligned} \tag{21}$$

The level sets of $V(\overline{r}, \overline{\theta})$ are closed (due to continuity), bounded in $\overline{\theta}$ (due to boundedness of the trigonometric nonlinearities), and radially unbounded in $\overline{r}$ (due to Assumption 1). Next, we investigate the time derivative of $V(\overline{r}, \overline{\theta})$ along the trajectories of the system, which is given by

$$\begin{aligned} \dot{V}(\overline{r}, \overline{\theta}) &= \left(\nabla_{\overline{r}_j} V\left(\overline{r}, \overline{\theta}\right)\right)^{\mathrm{T}} \dot{\overline{r}}_j + \left(\nabla_{\overline{\theta}_j} V\left(\overline{r}, \overline{\theta}\right)\right)^{\mathrm{T}} \dot{\overline{\theta}}_j \\ &= -\left(\nabla_{\overline{r}_j} V\left(\overline{r}, \overline{\theta}\right)\right)^2 - \overline{r}_j^2 \left(\frac{1}{\overline{r}_j^2} \nabla_{\overline{\theta}_j} V\left(\overline{r}, \overline{\theta}\right)\right)^2 \leq 0. \end{aligned}$$

Therefore, the sublevel sets of $V(\overline{r}, \overline{\theta})$ are forward invariant, and we conclude by LaSalle's invariance principle [29, Theorem 4.4] that the dynamics (12) converge to the largest positively invariant set contained in

$$\left\{ (\overline{r}, \overline{\theta}) \in \mathcal{I} :\, V(\overline{r}, \overline{\theta}) \leq V(\overline{r}_0, \overline{\theta}_0)\,,\, \dot{V}(\overline{r}, \overline{\theta}) = 0 \right\},$$

where we incorporated the positive invariance of $\mathcal{I}$. Next, we characterize the set of solutions that satisfy $\dot{V}(\overline{r}, \overline{\theta}) = 0$, i.e., the set of non-zero amplitude equilibria

$$\nabla_{\overline{r}_j} V\left(\overline{r}, \overline{\theta}\right) = 0;\; \frac{1}{\overline{r}_j^2} \nabla_{\overline{\theta}_j} V\left(\overline{r}, \overline{\theta}\right) = 0\,, \tag{22}$$

which, by using (12) and (20), can be compactly written as:

$$H + C\overline{r} = 0;\; S\overline{r} = 0\,, \tag{23}$$

where the entries of H, C and S are given by:

$$\begin{aligned} &\left[H\right]_j = \;\overline{h}(\overline{r}_j) - \tfrac{N}{2}\overline{r}_j \;\;,\; \left[C\right]_{j\ell} = \tfrac{1}{2}\cos(\overline{\theta}_{jl}), \\ &\left[S\right]_{j\ell} = \tfrac{1}{2}\sin(\overline{\theta}_{jl}). \end{aligned} \tag{24}$$

Notice that S is a null matrix when $\overline{\theta}_{jk} = m\pi\ \forall j,k \in \mathcal{N}, m \in \mathbb{Z}$. So, one set of solutions is represented by (18).

Now, let us consider the case when S is not a null matrix and establish that (17) describes such solutions. Observe that when S is not a null matrix, then S and C have the same null space over the field of reals. (See Proposition 2 in the Appendix.) Thus, (23) effectively reduces to:

$$H = 0,\; S\overline{r} = 0\,. \tag{25}$$

We note that $H = 0$ ensures that equilibrium radii are identical, given by ρ which satisfies

$$\overline{h}(\rho) = \frac{N}{2}\rho\,. \tag{26}$$

Incorporating identical radii, we get $S1_N = 0$ which implies $C1_N = 0$ (as they have identical null spaces) and therefore:

$$\mathrm{e}^{\jmath\overline{\theta}_j} \sum_{k=1}^{N} \left(\mathrm{e}^{-\jmath\overline{\theta}_k}\right) = 0\ \forall j\, \in \mathcal{N}, \tag{27}$$

which gives the phase-balanced state set $\mathcal{S}$. Thus, all such trajectories, where $\overline{\theta}_{jk} \neq m\pi, m \in \mathbb{Z},\ \forall j,k \in \mathcal{N}$, originating in $\mathcal{I}$ converge to $\mathcal{S}$. Thus, the dynamics either converge to $\mathcal{S}$ (17) or $\mathcal{S}'$ (18). ■

Recall that $\mathcal{S} \cap \mathcal{S}'$ is not necessarily empty. When the number of oscillators are even and the number of oscillators in each of phase-synchronized clusters in $\mathcal{S}'$ are equal then such equilibria belong to $\mathcal{S}$ as well. (See Fig. 1.) As we are interested in the asymptotic convergence to the phase-balanced states, we show next that the equilibria that belong to $\mathcal{S}'$ but not to $\mathcal{S}$ are locally unstable and thus almost all trajectories in $\mathcal{I}$ converge to the phase-balanced state set.

C. Local Instability of Solutions

Now, we focus on the set of solutions described by $\mathcal{S}'$ (18), of which the phase synchronized solutions, $\mathcal{S}''$ (19), are a special case and have been widely studied in the context of Liénard-type oscillators [19]. We show that our chosen feedback makes these equilibiria locally unstable if they do not lie in $\mathcal{S}$. The following theorem establishes the result.

Theorem 2. *Equilibria of* (12) *that reside in the set* $\mathcal{S}\backslash\mathcal{S}'$ *are unstable. Consequently, phase-synchronized solutions that lie in the set* $\mathcal{S}''$ *are also unstable.*

Proof. Linearizing (12) around the equilibria in $\mathcal{S}'$, we get:

$$J' = \left[\begin{array}{c|c} J'_{\rm A} & 0_N \\ \hline 0_N & J'_{\rm D} \end{array}\right]. \tag{28}$$

The entries of $J'_{\rm A}$, $J'_{\rm B}$, $J'_{\rm C}$, and $J'_{\rm D}$ are specified as:

$$[J'_{\rm A}]_{j\ell} = \begin{cases} \varepsilon\left(-\overline{h}'(\rho_j) + \frac{N-1}{2}\right) & \text{if } j = \ell \\ -\frac{\varepsilon}{2} & \text{if } j \neq l \text{ \& } \overline{\theta}_{j\ell} = 2m\pi \\ \frac{\varepsilon}{2} & \text{if } j \neq l \text{ \& } \overline{\theta}_{j\ell} = (2m+1)\pi \end{cases}$$

$$[J'_{\rm D}]_{j\ell} = \begin{cases} -\sum_{\ell=1,\ell\neq j}^{N}[J'_{\rm D}]_{j\ell} & \text{if } j = \ell \\ -\frac{\varepsilon}{2} & \text{if } j \neq l \text{ \& } \overline{\theta}_{j\ell} = 2m\pi \\ \frac{\varepsilon}{2} & \text{if } j \neq l \text{ \& } \overline{\theta}_{j\ell} = (2m+1)\pi \end{cases}$$

where $m \in \mathbb{Z}$ and ρ_j is the equilibrium radius for the jth oscillator. Since J' is block diagonal, therefore its eigenvalues are eigenvalues of $J'_{\rm A}$ and $J'_{\rm D}$. In the following, we focus the analysis on the definiteness properties of the symmetric submatrix $J'_{\rm D}$ associated with the angle dynamics. Recall that for the bi-cluster synchronous set, the oscillators belong to one of the two clusters on the circle (see Fig. 1) which differ in phase π (depending on the fact that the phase-differences are odd or even multiples of π). The subsequent analysis can be divided into three cases:

i) The sizes of the two clusters differ by more than one: The diagonal entries $J'_{\rm D}$ corresponding to the bigger cluster are positive, and since $e_j^{\rm T} J'_{\rm D} e_j > 0$ (j is index of the node in the bigger cluster), it is not negative semidefinite, therefore $J'_{\rm D}$ must have at least one positive eigenvalue [31]. Therefore, the solutions that lie in (18) such that the sizes of the synchronized clusters (that are π apart) differ by more than one are unstable.

ii) The sizes of the clusters differ by one: The diagonal entries are either 0 (for the nodes in the bigger cluster) or -2 (for the nodes in the smaller cluster). Thus, there exists a symmetric principal minor of order 2 (corresponding to two nodes in distinct clusters) of the form

$$\frac{\varepsilon}{2} \cdot \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & -2 \end{bmatrix}$$

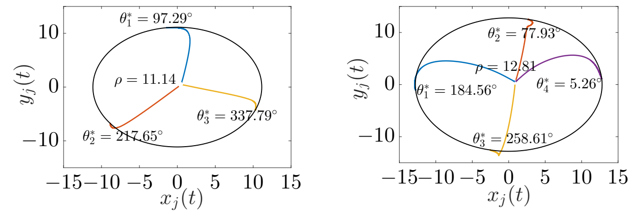
which features a positive eigenvalue. Therefore $J'_{\rm D}$ cannot be negative semi-definite [31].

iii) The size of both the clusters are the same: The solution also belongs to the phase-balanced set, $\mathcal{S}$. ■

Corollary 1 (Main result). *Consider the collective motion of the N networked oscillators, with the dynamics of each described by the flow* (12)*. Then, for almost all initial conditions $(\overline{r}_0, \overline{\theta}_0) \in \mathcal{I}$, the trajectories converge to the phase-balanced set $\mathcal{S}$* (17)*.*

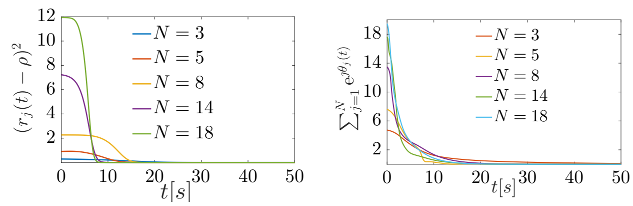
IV. Numerical Simulation Results

We consider a fully connected network of Van der Pol oscillators, which are a special class of Liénard type oscillators, with $h(x) = \varepsilon(\alpha x - \beta x^3)$, where $\alpha > 0$, $0 < \varepsilon << 1$ and $\beta > 0$ are real numbers. With parameters $\varepsilon = 0.1$, $\alpha = 0.1$ and $\beta = 0.1$ we simulate two cases to demonstrate the nature of the solutions. Figure 2 shows that for $N = 3$, a splay



(a) (b)

Fig. 2: a) A system of 3 fully-connected Van der Pol oscillators achieves the splay state (a particular case of the phase-balanced state $\mathcal{S}$ (17)). b) A system of 4 fully-connected Van der Pol oscillators achieves the phase-balanced state $\mathcal{S}$ (17) that does not lie in $\mathcal{S}'$.



(a) (b)

Fig. 3: Amplitude error, defined as the squares of deviation from the equilibrium, and phase error, defined as the sum of the complex exponentials of the deviations of the phase at a particular node with its neighbors decay to zero.

state is reached from arbitrary initial conditions, where the oscillators have the same radii $\rho = 21.9089$, and their phases are evenly spaced apart by $2\pi/3$. However, for $N = 4$, we achieve phases which are not evenly spread but satisfy the equilibrium condition for angles laid down in Theorem 1. To further validate the analysis, we plot the amplitude error and phase error for five different scenarios as shown in Fig.2. The radii settle down to ρ, as given by equation (26) in Theorem 1, and $\sum_{j=1}^{N} e^{\jmath(\theta_j)} = 0$.

V. Conclusion

We studied collective motion of identical Liénard-type oscillators with all-to-all weak coupling that follow second order dynamics of a weakly nonlinear Liénard system with a given frequency and showed that for a fully-connected network, the trajectories converge to the phase-balanced state. Furthermore, we demonstrated equilibria outside the phase-balanced set are locally unstable and therefore the trajectories converge to the phase-balanced state set for large set of initial conditions. We are working on leveraging this self-organizing phenomenon in networks of dc-dc converters for carrier wave interleaving for switching to minimize current ripple and harmonics. Furthermore, extending this to other graph topologies is also a part of ongoing investigations.

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Appendix

Proposition 2. *Consider the matrices C and S defined in (24). Further, assume that the S is not a null matrix, i.e.,$\overline{\theta}_{jk} \neq m\pi, \forall j,k \in \mathcal{N}, m \in \mathbb{Z}$. Then, the null spaces of C and S (over the field of reals) are identical.*

Proof. Define matrix $E = C + \jmath S$. Since E is a outer product of $z_0 z_0^{\mathrm{H}}$ (and therefore rank 1) where z_0 is a complex vector with entries $\mathrm{e}^{\jmath\overline{\theta}_j}$ and therefore $\mathrm{Ker}(E) = \mathrm{Ker}(z_0^{\mathrm{H}})$ (using the positivity of the inner product). Therefore,

$$\begin{aligned}\mathrm{Ker}_{\mathbb{R}}\left(E\right) &= \mathrm{Ker}_{\mathbb{R}}\left(z_0^H\right) \\ &= \left\{ v \in \mathbb{R}^N : z_0^{\mathrm{H}} v = 0 \right\} \\ &= \Bigg\{ [v_1, v_2, \ldots, v_N]^{\mathrm{T}} \in \mathbb{R}^N : \sum_{j=1}^{N} (\cos\overline{\theta}_j) v_j = 0\,, \\ &\qquad \sum_{j=1}^{N} (\sin\overline{\theta}_j) v_j = 0 \Bigg\}.\end{aligned}$$

This is the intersection of two hyperplanes in $\mathbb{R}^N$. Since we have excluded the cases when $\overline{\theta}_{jk} = m\pi, m \in \mathbb{Z}$, the hyperplanes are distinct (hyperplanes are coincident when $\tan\overline{\theta}_j = \tan\overline{\theta}_k \forall j,k \in \mathcal{N}$) and thus $\mathrm{Ker}_{\mathbb{R}}(E)$ is an $(N-2)$-dimensional real subspace of $\mathbb{R}^N$. From $E = C + \jmath S$, we obtain $\mathrm{Ker}_{\mathbb{R}}(E) = \mathrm{Ker}_{\mathbb{R}}(C) \cap \mathrm{Ker}_{\mathbb{R}}(S)$. If all the three kernels involved are $N-2$-dimensional, then this yields that $\mathrm{Ker}_{\mathbb{R}}(E) = \mathrm{Ker}_{\mathbb{R}}(C) = \mathrm{Ker}_{\mathbb{R}}(S)$. By using the fact that for square matrices, X and Y, $\mathrm{rank}(X+Y) \leq \mathrm{rank}(X) + \mathrm{rank}(Y)$, we can conclude that $S = (E - E^{\mathrm{H}})/2\jmath$ and $C = (E + E^{\mathrm{H}})/2\jmath$ have rank at most 2 ; to show the three kernels involved are $N-2$-dimensional, it suffices to show that $\mathrm{rank}(C)$ and $\mathrm{rank}(S)$ are greater than 2. We can pick $m < n$ satisfying $\sin(\overline{\theta}_{mn}) \neq 0$ (by assumption) and then observe that the determinants of $(\{m,n\}, \{m,n\})$-minors of C and S are greater than zero. Since the rank of a matrix is the largest order of any non-zero minor and we have a second order minor which is nonzero, the rank is at least 2. Thus, the nullspace of C and S are identical when S is not a null matrix. ■

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