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Synchronization of Liénard-type Oscillators in Heterogenous Electrical Networks

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***Abstract*— Motivated by potential applications for power-electronic converters in microgrids, we study the problem of global asymptotic synchronization of Liénard-type nonlinear oscillators in heterogenous LTI electrical networks with series R-L circuits modeling interconnections. By heterogeneous, we mean that the resistance-to-inductance ratios of the lines are not all the same. Building on our previous work, we derive sufficient conditions for global asymptotic synchronization by using a first-order filter on the outputs of the oscillators. Our approach leverages a coordinate transformation to a system that emphasizes signal differences and the use of passivity based arguments to establish synchronization of the network of oscillators asymptotically. The analysis subsumes and generalizes previous conditions that have been derived for Liénard oscillators in homogeneous electrical networks. Numerical simulations are provided to validate the results.**

I. Introduction

This paper presents a set of conditions for global asymptotic synchronization of identical Liénard-type circuits coupled through a class of passive electrical LTI networks with arbitrary topologies where the interconnecting lines are modeled as series R-L circuits. The problem setup is motivated by applications of controlling power-electronic converters in microgrids. For instance, utilizing Liénard oscillators to guarantee decentralized interleaving between units of parallel-connected converters in dc microgrids [1] presents a compelling alternative in modular plug-n-play settings or regulating inverters to emulate the dynamics of limit-cycle oscillators is an effective strategy to realize a stable power system as synchrony emerges without any external communication and feedback is at faster-than-AC time scales [2]–[4].

Typically, the analysis in power networks assumes lines have the same resistance-to-inductance ratios, which may not be the case in reality. We focus on the case of heterogeneous ratios, and in so doing generalize the synchronization conditions of our previously proposed control schemes in [4], [5]. Additionally, synchronization of Liénard-type oscillators is of interest to a wide variety of fields and has been explored in detail [6]–[12]. Therefore, the analysis setup can benefit other circuits-related applications involving, e.g., solid-state circuit oscillators, microwave oscillator arrays and semiconductor laser arrays [6], [13], [14].

This work extends our previous effort in [15] where we made use of the uniform line assumption (ratio of resistance-to-inductance in every line is assumed constant) to establish global asymptotic synchronization for Liénard oscillators. In a similar vein, we construct a differential system (also referred to as an incremental system in [16], [17]) emphasizing signal differences; construct a quadratic Lyapunov function and derive conditions for it to be decreasing. We arrive at a synchronization condition that is related to the algebraic connectivity of the network and establishes *internal stability* in the *heterogenous case* thereby generalizing our previous results. Additionally, similar to the approach in [15], [18], we also make use of Kron reduction [19] to extend our analysis to a wider class of network topologies. As remarked in [15], unlike the efforts on analyzing synchronization of coupled Liénard oscillators which resorted to averaging techniques [5], [20], which is only valid in a quasi-harmonic regime, i.e., when the limit-cycles are almost circular, we make no assumptions of this sort in the present work, and our results hold for non-harmonic limit cycles as well.

Our work, in particular, is inspired by Lyapunov- and passivity-based methods [17], [21]–[25]. Input-output $\mathcal{L}_2$ methods to address synchronization problems have been proposed in [16], [18]. The adopted analysis approach allows us to relax a Lipschitz condition on the oscillation-inducing nonlinearity [18]. Furthermore, it overcomes a limitation of the modeling-and-analysis framework in [24]–[26] requiring the edges (the lines connecting the nonlinear circuits) to be input strictly passive, e.g., a parallel R-L connection. The analysis presented in this work allows the edges to be series R-L connections which are output strictly passive and more pertinent in the context of power networks.

The remainder of this paper is organized as follows. Section II introduces mathematical notation and pertinent preliminaries focused particularly on the oscillator dynamical model and the network. In Section III, we establish global asymptotic synchronization conditions for the system of coupled circuits leveraging Lyapunov stability theory. Simulations are provided in Section IV to validate the approach. We conclude the paper in Section V.

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II. PRELIMINARIES

In this section, we begin by establishing notation and then provide a brief background about the Liénard oscillator model and the electrical network.

A. Notation

Given a real-valued N-tuple $\{u_1, \ldots, u_N\}$ denote the corresponding column vector as $u = [u_1, \ldots, u_N]^{\mathrm{T}}$, where $(\cdot)^{\mathrm{T}}$ denotes the matrix transpose. Denote the $N \times N$ identity matrix as I_N and the N-dimensional vector of all ones as 1_N. Signal differences will be quantified by the so-called *projector matrix* [16], [17]

$$\Pi := I_N - \frac{1}{N} 1_N 1_N^{\mathrm{T}} . \tag{1}$$

For a vector u, define $\widetilde{u} := \Pi u$ to be the corresponding *differential* vector. Also, for the vector u, $\dot{u} := [\frac{du_1}{dt}, \ldots, \frac{du_N}{dt}]^{\mathrm{T}}$ denotes the vector with element-wise time derivatives and $\mathrm{diag}\{u\}$ denotes a diagonal matrix which has elements of u stacked along the main diagonal.

B. Liénard-type Oscillators

Liénard's equation is a nonlinear second-order differential equation of the general form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{2}$$

This dynamical model is commonly employed to study oscillations in nonlinear dynamical systems. For instance, the ubiquitous Van der Pol oscillator dynamics can be recovered as a special case of (2) [6]. The following theorem establishes conditions that the functions $f(\cdot)$ and $g(\cdot)$ need to satisfy so that the system (2) exhibits a unique and stable limit cycle around the origin.

Theorem 1 (Liénard's Theorem [27]). *Consider the dynamical system* (2). *Assume that functions* $f(x)$ *and* $g(x)$ *satisfy the following:*

(A1) $f(x)$ *and* $g(x)$ *are continuously differentiable;*
(A2) $g(x) > 0, \forall x > 0$*; and* $g(x)$ *is an odd function;*
(A3) $f(x)$ *is an even function;*
(A4) The odd function $h(x) := \int_0^x f(\tau)d\tau$ *has exactly one positive zero at* $x = \zeta$*, is strictly negative for* $0 < x < \zeta$*, is positive and nondecreasing for* $x > \zeta$*, and* $\lim_{x\to\infty} h(x) = \infty$*.*

Then, the system (2) *has a unique and asymptotically stable limit cycle surrounding the origin in the phase plane.*

In this work, we focus on *forced* Liénard-type oscillator circuits that exhibit unforced oscillations at the frequency ω. We will now describe pertinent models with a circuit-theoretic interpretation of (2). The terminal voltage, v, of the oscillators we study is governed by:

$$\ddot{v} + f(v)\dot{v} + \omega^2 v = \dot{u}\,, \tag{3}$$

where $f(\cdot)$ satisfies the conditions in Theorem 1, the function $g(\cdot)$ takes the particular form $g(v) = \omega^2 v$, and u is the input current to the oscillator. Figure 1(a) depicts an example of the

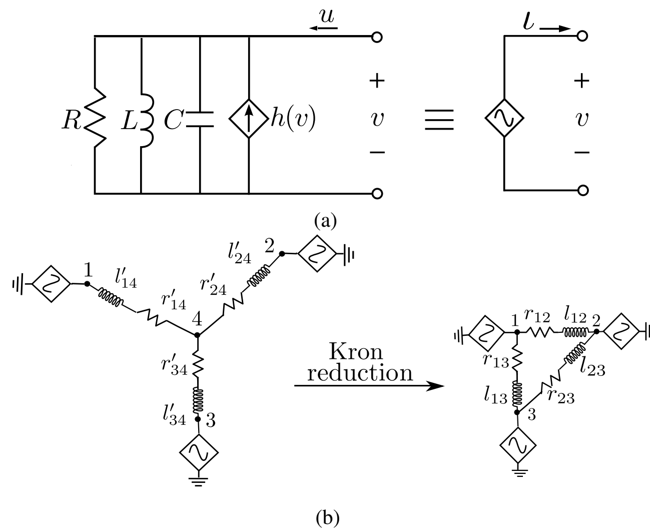

Fig. 1: a) The Van der Pol oscillator admits the dynamics in (3) with $\omega = 1/\sqrt{LC}$, $\varepsilon = \sqrt{L/C}$ and $f(v) = \varepsilon\alpha\omega(\beta v^2 - 1)$, where α and β are positive constants. The nonlinear voltage-dependent current source is denoted by $h(v) := \int f(v)dv$. b) Consider a collection of nonlinear oscillators as (a) connected in a network with no shunts. The Kron-reduced network also has no shunts.

well-known Van der Pol circuit which falls under this class of nonlinear oscillators. To simplify illustration, when the Van der Pol oscillator circuit is coupled to an electrical network, we will use the shorthand depicted in the right in Fig. 1(a) to denote the nonlinear oscillator circuit. Furthermore, we will denote the output current of the circuit by ι and note that $u = -\iota$. (We denote currents by the lower-case Greek letter, iota, ι, and the time rate of change of the current by $\dot{\iota}$.)

Next, we present a key lemma which establishes that the function $f(\cdot)$ in Liénard's equation (2) cannot be unbounded from below and attains a finite lower bound. We will find this result useful when we derive the synchronization condition for the network of Liénard circuits as this property renders the oscillator model incrementally output feedback passive as observed in [24], [25].

Lemma 1. *Consider the function* $f(x)$ *that satisfies the conditions in Theorem 1. There exists a lower bound* $\rho > 0$ *such that* $f(x) \geq -\rho$*.*

Proof. See [15, Lemma 1]. $\square$

C. Electrical Network

The nonlinear circuits are coupled through a connected and passive LTI network where the interconnecting lines are modeled as series R-L circuits. We focus on R-L networks with heterogeneous line characteristics without shunt elements thereby relaxing the uniform resistance-to-inductance ratio modeling assumption that is commonly used in the literature in similar settings. Some of the nodes of the electrical network are connected to the nonlinear circuits and the rest have zero current injections. Notice that the nodes with zero current injections can be eliminated to obtain an equivalent circuit with elementary algebraic manipulations of the network admittance matrix. This model reduction procedure is called Kron reduction. For instance, with reference to Fig. 1(b), notice that node 4 in the Y network (left) has zero current injection and, therefore, can be eliminated to yield an electrically equivalent Δ network (right). Further details on Kron reduction can be found in [19]. It can be shown that Kron reduction of networks without shunt elements yields networks without shunt elements. Therefore, it suffices to consider only the Kron-reduced networks (i.e., networks with no nodes having zero current injections) in subsequent developments while recognizing that the originating networks may have had more complex topologies.

Lemma 2. *The following statements are equivalent: i) The original electrical network has no shunt elements; ii) The Kron-reduced network has no shunt elements.*

Proof. See [18, Theorem 1]. □

Now that we have introduced the basic properties of the oscillators and networks that are relevant to our study, we set up the problem of global asymptotic synchronization for the coupled network of oscillators. In particular, we design a first-order filter on the outputs of the oscillators to ensure synchronization which can otherwise not be guaranteed with just the electrical interconnection. Furtheremore, the analysis suggests a class of heterogeneous dynamic links between the identical oscillators that guarantee synchronization and adds to the findings in [28], [29] where dynamic feedback for Lur'e systems was designed to guarantee synchronization.

III. Synchronization of Liénard-type Oscillators

Consider a Kron-reduced network interconnecting the N Liénard-type circuits (without shunt elements), recognizing that the originating network may have had additional nodes that were eliminated. Let $\mathcal{N} = \{1, \ldots, N\}$ denote the set of nodes with the oscillators, and let $\mathcal{L} = \{1, \ldots, L\}$ denote the set of interconnecting transmission lines (edges in the underlying graph). Furthermore, let r_ℓ and l_ℓ denote the resistance and inductance of the ℓ-th edge. To capture the different resistance-to-inductance ratios in branches of the electrical network, define

$$\frac{r_\ell}{l_\ell} =: \gamma_\ell \quad \forall\, \ell \,\in\, \mathcal{L}\,. \tag{4}$$

Furthermore, let $v = [v_1, \ldots, v_N]^{\mathrm{T}}$ collect the voltages at the oscillator terminals, $\eta = [\eta_1, \ldots, \eta_L]^{\mathrm{T}}$ collect the edge currents and $\iota = [\iota_1, \ldots, \iota_N]^{\mathrm{T}}$ denote the vector with current injections as its entries. With this notation in place, we first establish the dynamics of pertinent voltage and current states in the network.

A. Network Dynamics

Kirchoff's voltage law for the ℓ-th branch in the Kron-reduced network between the j-th and k-th oscillator yields

$$r_\ell \eta_\ell + l_\ell \dot{\eta}_\ell = v_j - v_k\,, \tag{5}$$

where η_ℓ is the branch current in the ℓ-th branch. Rearranging terms, and taking the Laplace transform yields

$$(s + \gamma_\ell)\eta_\ell = \frac{1}{l_\ell}(v_j - v_k) = \frac{1}{l_\ell} w_\ell\,, \tag{6}$$

where w_ℓ denotes the voltage difference across the ℓ-th edge. It follows that the vector of all nodal-voltage differences can be expressed as

$$w = B^{\mathrm{T}} v, \tag{7}$$

where B is the edge-oriented incidence matrix of the network. From (6) and (7), the Laplace transform of the nodal current-injection vector can be expressed as

$$\iota = B\eta = B D_\gamma(s) D_l^{-1} B^{\mathrm{T}} v, \tag{8}$$

where we define the shorthands

$$D_\gamma(s) := \mathrm{diag}\{(s+\gamma_1)^{-1}, \ldots, (s+\gamma_L)^{-1}\}, \tag{9}$$
$$D_l := \mathrm{diag}\{l_1, \ldots, l_L\}. \tag{10}$$

The (transfer) matrices (9) and (10) are evidently a linear mapping describing the electrical network part of the system.

Recall that the current injections at the nodes are used as feedback to the oscillators to close the loop. In particular, the entries of the vector u (i.e., the vector of oscillator input currents) are specified as $u = -\iota$. (See also, Fig. 1.) While this electrical interconnection has been shown to be sufficient for synchronization in many cases [4], [17], we will find it inadequate in the present heterogeneous setting.

B. Oscillator Dynamics

Having described the network dynamics, we now establish the dynamical model for the oscillators. We do so as a *differential system* (leveraging the projector-matrix notion), while parameterizing the system in the conventional (ξ, v) Lur'e coordinates, where $\xi_j := \omega \int_0^t v_j(t)dt$. Thus, the collective *differential* oscillator dynamics (3) are

$$\begin{aligned} \Pi\dot{\xi} &= \omega\Pi v, \\ \Pi\dot{v} &= -\omega\Pi\xi - \Pi H(v) + \Pi u\,, \end{aligned} \tag{11}$$

where $v \in \mathbb{R}^N$ is the vector of terminal voltages, $\xi = [\xi_1, \ldots, \xi_N]^{\mathrm{T}} \in \mathbb{R}^N$, Π is the projector (1), and we define the following function to contain the nonlinearities:

$$H(v) := \left[\int_0^{v_1} f(r)dr, \ldots, \int_0^{v_N} f(r)dr\right]^{\mathrm{T}}. \tag{12}$$

The block diagram in Fig. 2 captures the interconnection of the dynamical models discussed above for a Kron-reduced network of N identical Liénard-type oscillators coupled through a heterogeneous R-L network without shunt elements and $L = \binom{N}{2}$ edges. By way of notation, Δ denotes the nonlinear oscillator sub-system as specified by (11).

C. Standard Lyapunov Function

Relying solely on the electrical interconnection (i.e., $u = -\iota$) turns out to be inadequate for synchronization, and we observe this in simulations. Let us motivate this inadequacy

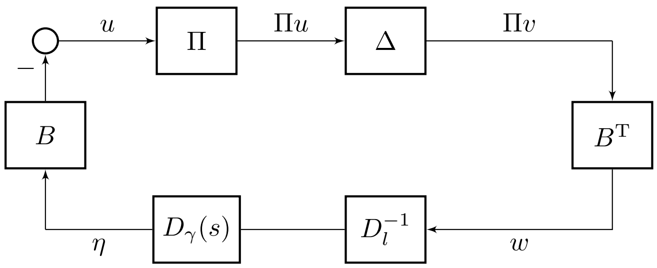

Fig. 2: Block-diagram representation of interconnected system without the filter and only current feedback.

by analytic reasoning. Consider the standard energy-based Lyapunov function suggested by circuit and passivity theory:

$$V = \tfrac{1}{2}(\Pi\xi)^{\mathrm{T}}(\Pi\xi) + \tfrac{1}{2}(\Pi v)^{\mathrm{T}}(\Pi v) + \tfrac{1}{2}\eta^{\mathrm{T}} D_l \eta\,, \tag{13}$$

where D_l is defined as in (10) The associated time derivative along trajectories of the differential oscillator and circuit dynamics (11), (5) with the interconnections $w = B^{\mathrm{T}}v$, $\iota = B\eta$, and $u = -\iota$ is (after some manipulations) obtained as

$$\dot V = -(\Pi v)^{\mathrm{T}}\Pi H(v) - \eta^{\mathrm{T}} D_r^{-1}\eta\,, \tag{14}$$

where $D_r = \mathrm{diag}\{r_1,\dots,r_L\}$. Observe that $\dot V$ is generally sign-indefinite. Alternatively, one may consider a synchronization-inspired Lyapunov function [15]

$$V = \tfrac{1}{2}(\Pi\xi)^{\mathrm{T}}(\Pi\xi) + \tfrac{1}{2}(\Pi v)^{\mathrm{T}}(\Pi v) + \tfrac{1}{2}(\Pi\iota)^{\mathrm{T}}(\Pi\iota). \tag{15}$$

The time derivative is obtained (after some manipulations) as

$$\dot V = -(\Pi v)^{\mathrm{T}}\Pi H(v) - \eta^{\mathrm{T}}B^{\mathrm{T}}B\Gamma\eta + \eta^{\mathrm{T}}B^{\mathrm{T}}L_l v\,, \tag{16}$$

where D_l is as defined in (10), and

$$L_l := BD_l^{-1}B^{\mathrm{T}}, \tag{17}$$

is the weighted Laplacian matrix associated with the susceptances (inverse inductances) of the network, and we defined

$$\Gamma := \mathrm{diag}\{\gamma_1,\dots,\gamma_L\} = \mathrm{diag}\{r_1/l_1,\dots,r_L/l_L\}. \tag{18}$$

Again, the time-derivative is indefinite, and no stability certificate can be established. We remark that these failing analysis approaches do not only show the inadequacy of the considered Lyapunov functions, but also confirm numerical observations: the closed-loop system may not synchronize.

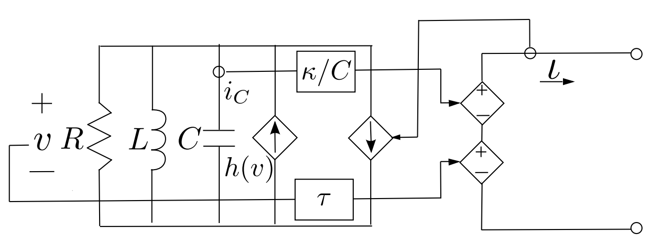

Fig. 3: The output filter is implemented by using the capacitor current and voltage which in turn control voltage sources.

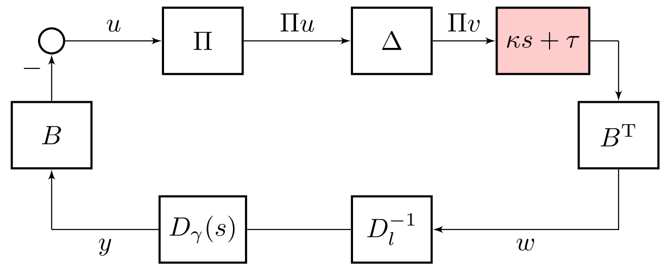

Fig. 4: Block diagram of the system with the proposed filter (shaded in red) placed at the outputs of the oscillator terminals.

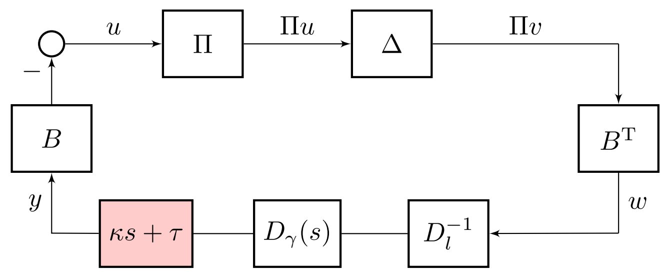

Fig. 5: Revised block diagram that facilitates analysis derived from the one in Fig. 4 leveraging the commutativity of the filter and B.

D. *Filters for Synchronization*

We now propose a scalar dynamic filter $\kappa s + \tau$ for the output of the oscillators to facilitate synchronization in the heterogeneous network. A circuit diagram of the implementation is shown in Fig. 3. The networked-dynamics of the oscillators with this output filter are shown in the block diagram in Fig. 4. Note that the cascade system of the second-order nonlinear oscillator (11) and the first-order filter is causal. Hence, from an implementation point of view for converter control [1], [4], the nonlinear oscillator with output filter can readily be implemented in software. For other potential circuit-related applications, it can be realized with current-controlled and voltage-controlled voltage sources.

Since the filter described by the scalar transfer-function block $\kappa s+\tau$, commutes with all other linear blocks in Fig. 4, we can study the equivalent block-diagram in Fig. 5 for the sake of a simplified analysis. The interpretation of Fig. 5 is that the edge currents are filtered by the system $\kappa s + \tau$, where κ and τ are design parameters that we can select to ensure stability of the interconnection. The following result establishes sufficient conditions for the design of this filter so that global asymptotic synchronization in the network of oscillators can be guaranteed.

Theorem 2. *Consider a network of N identical Liénard-type circuits with unforced frequency ω connected through an output filter $\kappa s+\tau$ to a heterogeneous electrical network. Let the maximum resistance-to-inductance ratio in the electrical network be $\gamma_{\max}$, and L_l be the Laplacian matrix associated with the inductances* (17). *Denote the algebraic connectivity corresponding to L_l by $\lambda_2(L_l)$, and let $-\rho$ be the global minimum for the function $f(\cdot)$ for each Liénard oscillator. If* *the parameters of the output filter, $\kappa s + \tau$, satisfy*

$$\kappa \cdot \lambda_2(L_l) > \rho, \quad \tau/\kappa > \gamma_{\max}, \tag{19}$$

then the terminal voltages of the oscillators in the network synchronize asymptotically.

Proof. Consider the block diagram in Fig. 5 which illustrates the networked connection of the oscillators with the output filters (shaded red). In particular, our approach hinges on exploiting the passivity properties of the subsystems and then considering a composite Lyapunov function to establish synchronization. We begin by considering the oscillator system described by (11) and denoted by Δ in Fig. 5.

Nonlinear Subsystem. The nonlinear block, Δ, is *incrementally output-feedback passive*. To prove this, consider the following Lyapunov function for the system (11):

$$V_{\text{osc}} = \frac{1}{2}(\Pi\xi)^{\mathrm{T}}(\Pi\xi) + \frac{1}{2}(\Pi v)^{\mathrm{T}}(\Pi v). \tag{20}$$

Given the dynamics (11), we get:

$$\begin{aligned}\dot{V}_{\text{osc}} &= -(\Pi v)^{\mathrm{T}}\Pi H(v) + (\Pi v)^{\mathrm{T}}\Pi u \\ &\leq \rho(\Pi v)^{T}(\Pi v) + (\Pi v)^{\mathrm{T}}\Pi u\,, \end{aligned} \tag{21}$$

where the last line leverages the lower-boundedness of f, i.e., $f(r) \geq -\rho \forall r \in \mathbb{R}$ as established in Lemma 1.

Linear Subsystem. The transfer function of the linear subsystem which establishes the mapping between the edge voltage differences and filtered edge currents (i.e., it captures the dynamics of the R-L line dynamics including the designed filter) is given by

$$D(s) = (\kappa s + \tau)D_\gamma(s)D_l^{-1}, \tag{22}$$

where, $D_\gamma(s)$ and D_l are defined in (9) and (10), respectively. (See Fig. 5.) The state-space model of this system can be realized as

$$\begin{aligned}\dot{x} &= -\Gamma x + w \\ y &= D_l^{-1}\left((\tau I_L - \kappa\Gamma)x + \kappa w\right)\,, \end{aligned} \tag{23}$$

where Γ is defined in (18), and I_L is the identity matrix of size $L \times L$. For these dynamics, define the Lyapunov function

$$V_{\text{net}} = \frac{1}{2}x^{\mathrm{T}}Ax, \tag{24}$$

where $A := D_l^{-1}(\tau I_L - \kappa\Gamma)$ is a diagonal matrix with

$$A_{jj} := (\tau - \kappa\gamma_j) \cdot l_j > 0, \tag{25}$$

given that $\tau > \kappa\gamma_{\max}$ by assumption (19). Then, we see that

$$\begin{aligned}\dot{V}_{\text{net}} &= x^{\mathrm{T}}A(-\Gamma x + w) && \text{(26a)}\\ &= -x^{\mathrm{T}}A\Gamma x + w^{\mathrm{T}}(y - \kappa D_l^{-1}w) && \text{(26b)}\\ &\leq w^{\mathrm{T}}y - \kappa w^{\mathrm{T}}D_l^{-1}w\,, && \text{(26c)}\end{aligned}$$

where we substitute $Ax = (y - \kappa D_l^{-1}w)$ in (26a) to get (26b) and subsequently use positive definiteness of the matrix $A\Gamma$ to obtain (26c). Thus, the linear subsystem is *input strictly passive*.

Interconnection of Linear and Nonlinear Subsystems. The outputs of both the subsystems are connected via the edge-incidence matrix B as $w = B^{\mathrm{T}}v$ and $u = -By$. For the interconnected system, consider the storage function $V_{\text{sys}} = V_{\text{osc}} + V_{\text{net}}$ with derivative given by

$$\begin{aligned}\dot{V}_{\text{sys}} &= \dot{V}_{\text{osc}} + \dot{V}_{\text{net}} && \text{(27a)}\\ &\leq \rho(\Pi v)^{\mathrm{T}}(\Pi v) + (\Pi v)^{\mathrm{T}}\Pi u + w^{\mathrm{T}}y - \kappa w^{\mathrm{T}}D_l^{-1}w && \text{(27b)}\\ &= \rho(\Pi v)^{\mathrm{T}}(\Pi v) - v^{\mathrm{T}}By + v^{\mathrm{T}}By - \kappa v^{\mathrm{T}}BD_l^{-1}B^{\mathrm{T}}v && \text{(27c)}\\ &= \rho(\Pi v)^{\mathrm{T}}(\Pi v) - \kappa v^{\mathrm{T}}L_l v && \text{(27d)}\\ &\leq -\left(\kappa\lambda_2(L_l) - \rho\right)(\Pi v)^{\mathrm{T}}(\Pi v). && \text{(27e)}\end{aligned}$$

Above, (27b) follows from (26) and (21), and subsequently we used $u = -By$, $w = B^{\mathrm{T}}v$, $\Pi B = B$ to get (27c). Finally, Πv is orthogonal to 1_N spanning the nullspace of L_l, and thus

$$(\Pi v)^{\mathrm{T}}L_l(\Pi v) \geq \lambda_2(L_l)(\Pi v)^{\mathrm{T}}(\Pi v)\,. \tag{28}$$

Therefore, if $\kappa\lambda_2(L_l) > \rho$ then V_{net} is negative semidefinite and, therefore, by Lyapunov stability theorem and LaSalle's invariance principle [30, Theorem 4.1 and 4.4], the origin of the coupled system is globally asymptotically stable. This further implies that the network of oscillators synchronizes asymptotically. $\square$

We remark that, in contrast to our previous result [15], Theorem 2 guarantees *internal stability* of the overall system as we consider the branch states as well. The main stability condition (19) can always be met by choosing appropriate filter parameters κ and τ, and it has the following interpretation: the first inequality $\tau/\kappa > \gamma_{\max}$ restricts the filter time constant τ/κ relative to the largest network time constant $\gamma_{\max} = \max_\ell \frac{r_\ell}{l_\ell}$. Indeed, this first inequality allows us to investigate heterogeneous line dynamics by dominating their time-constants via the filter time constant τ/κ. The second inequality $\kappa \cdot \lambda_2(L_l) > \rho$ requires the effective network coupling (product of the filter gain κ and the algebraic connectivity $\lambda_2(L_l)$) to be larger than the (instability-inducing) negative damping bound ρ.

IV. Simulation Case Study

We consider a simulation case study of virtual oscillator controlled inverters in a microgrid to demonstrate the application of our study. Recall that in virtual oscillator control, states of the Lienard-type oscillators are used to generate sine PWM for the voltage source inverter which is then filtered through a low pass filter to remove the switching harmonics. In our example, a system of 3 parallel-connected voltage source inverters, each satisfying an identical local resistive load, is studied for the case when the output low-pass filters of the inverters are heterogeneous. The dynamics of the oscillators are described by (3) with $f(v) = \varepsilon(\beta v^2 - 1)$, from which it follows that $\rho = \varepsilon$. For the simulations, we set $\omega = 2\pi 60\,\mathrm{rad/s}$, $\beta = 1$ and $\varepsilon = 0.01$. Figure 6 shows

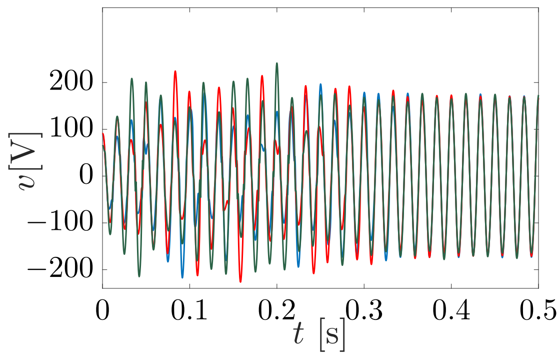

Fig. 6: A parallel network of 3 virtual oscillator controlled inverters, each satisfying an identical resistive load of $50\,\Omega$, have heterogeneous output low-pass R-L filters that take random values within specified tolerance of 10% of the nominal value. The filters for the three inverters have nominal values $1\,\Omega$, $1.5\,\Omega$ and $.5\,\Omega$ for the resistances; and $6\,\mathrm{mH}$, $9\mathrm{mH}$, and $3\mathrm{mH}$ for the inductances Initially, no filter is presented in the system and the system does not synchronize. At $t = 0.25\,\mathrm{sec}$, $\kappa s + \tau$ filter is introduced which satisfies the conditions of Theorem 2 and the system synchronizes.

synchronization with the designed filter $\kappa s + \tau$ that satisfies the condition of Theorem 2.

V. Conclusions

We extended the analysis in [15] to derive sufficient conditions for global asymptotic synchronization of identical Liénard-type circuits to a class of heterogeneous electrical RL networks where the inductance-to-resistance ratios could be different. Our analysis was based on passivity arguments – the Liénard-type circuits are incrementally output-feedback passive – and it outlined the design of first-order output filters for the circuits to ensure synchronization when connected in heterogeneous RL networks. Like the previous results in [15], [24], [25], we arrived at a condition on algebraic connectivity of the network and the filter parameters. As a part of future work, we would like to generalize this set of results to incorporate networks with shunt elements.

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